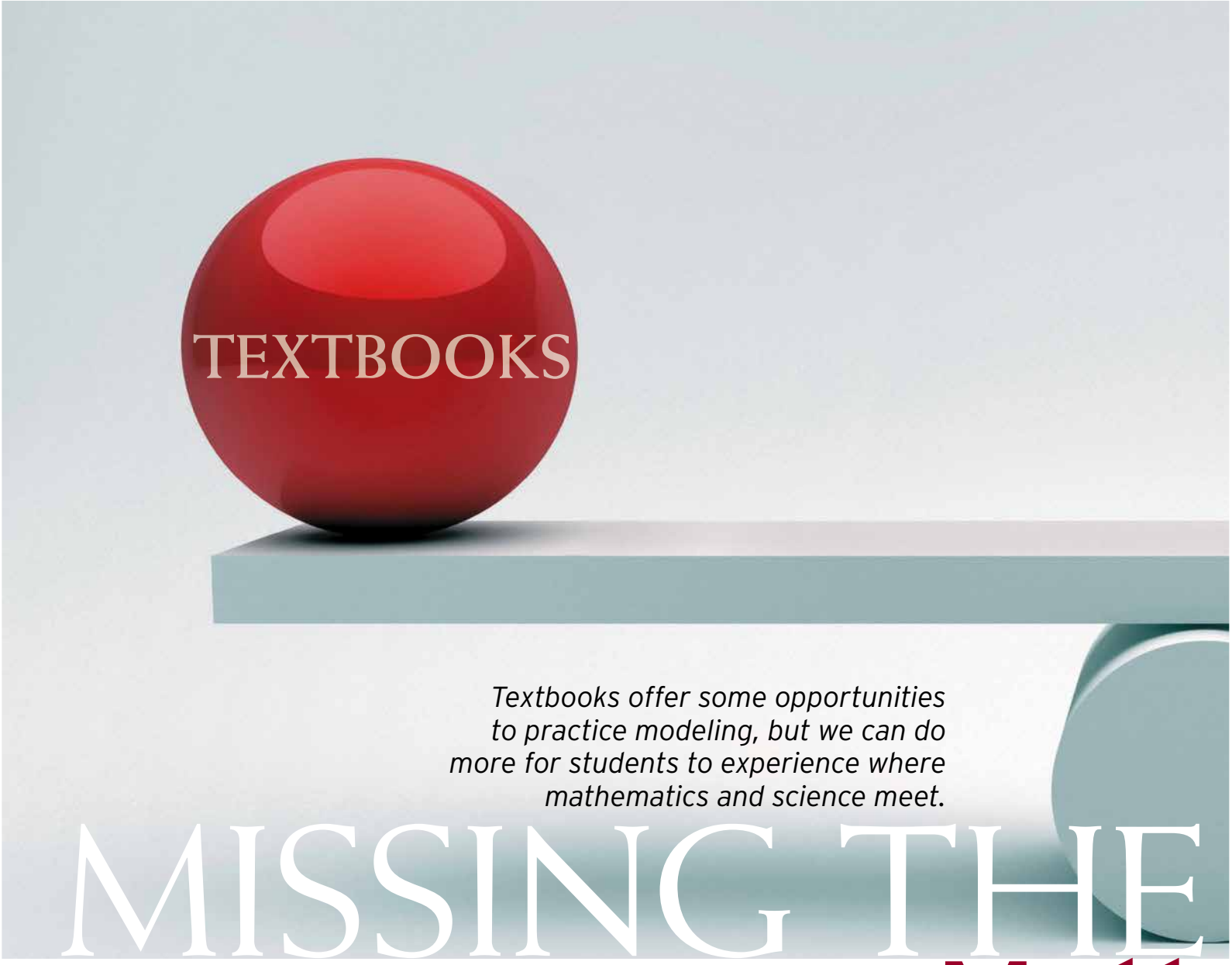

TEXTBOOKS

Textbooks offer some opportunities to practice modeling, but we can do more for students to experience where mathematics and science meet.

MISSING THE Math

Dan Meyer

The Common Core State Standards for Mathematics (CCSSM) have exerted enormous pressure on every participant in a child's education. Students are struggling to meet new standards for mathematics learning, and parents are struggling to understand how to help them. Teachers are growing in their capacity to develop new mathematical competencies, and administrators are growing in their capacity to support them.

These standards have also exerted pressure on textbook publishers, who must provide curriculum that aligns with the CCSSM. The CCSSM have made some of this existing content obsolete or pushed it to other grade levels. In other cases, publishers have had to develop new content aligned to the CCSSM. But a recent study of fourth-grade textbooks found that this alignment has been slippery, with many textbooks including content external to the CCSSM, failing to include critical CCSSM content or duplicating their previous unaligned editions to an inappropriate degree (Polikoff 2014).

This situation should concern us all given the large sums of money spent nationally on textbooks and the high degree to which teachers take their instructional cues from textbooks. What incentives do publishers have to undertake these costly alignments and developments? The CCSS issued a Publishers' Criteria, but these criteria are not binding

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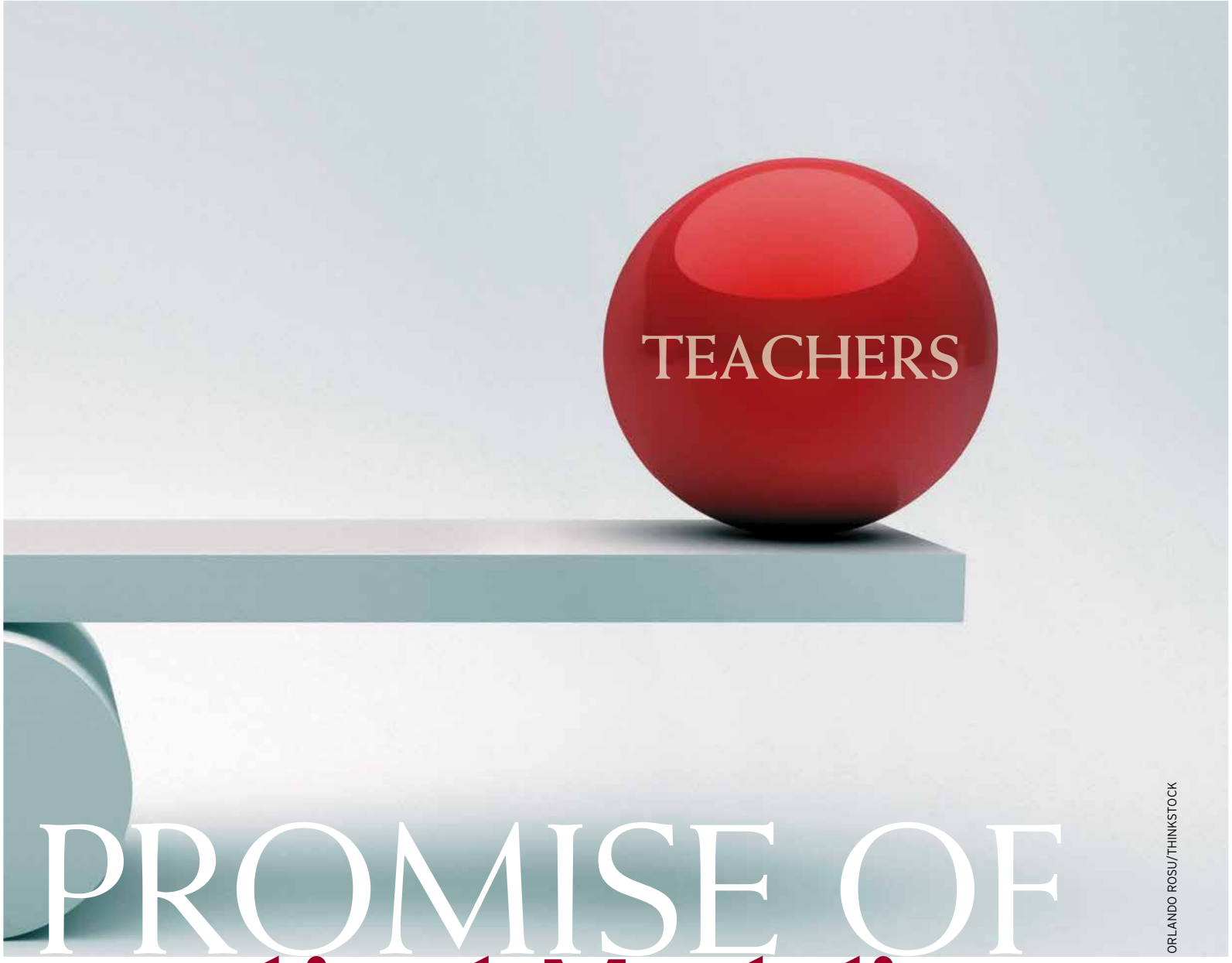

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ematical Modeling

in any sense. We—the people who buy textbooks or influence those who do—are publishers' only incentive.

With that rationale in mind, what follows is my analysis of how well textbooks fulfill the promise of one particular standard—mathematical modeling—as it is represented in the CCSSM.

I choose to examine modeling for several reasons. First, speaking strictly personally, I studied mathematics as a child and mathematics education as an adult because of powerful experiences I had using mathematics as a model for the world around me. I want students to have similar experiences. Second, in my work with teachers in professional development, I find modeling with mathematics (Standard for Mathematical Practice 4, CCSSI 2010, p. 7) to be one of the practice standards most in need of explication. Five different teachers may have five different understandings of its meaning. Third, mathematical modeling is the standard where mathematics and science meet. The practice standards of the Next Generation Science Standards (NSTA 2012) resemble the mathematical modeling standards of the CCSSM so closely that we should ensure that we get our end right.

In my research, I analyzed two textbooks in particular—an algebra 1 textbook and a geometry textbook, both published by McGraw-Hill (Carter et al. 2013a, 2013b). I chose these particular texts

because, whereas other publishers might simply identify a general CCSSM alignment in their textbooks, these two textbooks made the specific claim that their tasks captured specific standards such as "modeling." It seemed fair then to ask, "How well?" Through my analysis, I identified eighty-three tasks across the two years of mathematics that claimed to represent "modeling with mathematics."

MODELING IN THE CCSSM

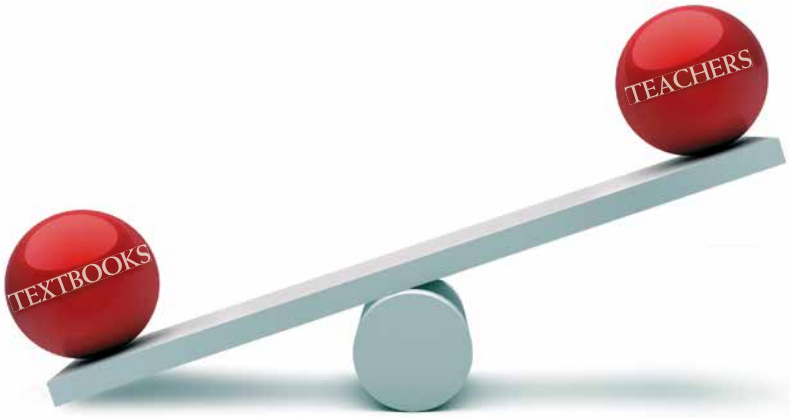
The CCSSM high school modeling standard describes five different actions that students take over the course of a complete modeling task:

1. Identifying essential variables in a situation
2. Formulating models from those variables
3. Performing operations using those models
4. Interpreting the results of those operations
5. Validating the conclusions of those results (CCSSI 2010, p. 72)

The CCSSM describe a sixth modeling action—reporting on the conclusions (p. 73)—that I chose not to consider in this analysis because it seemed within the classroom teacher's locus of control more than the textbook's. For any given modeling task, a teacher may easily request that students report their conclusions in the format of an advisory letter, a written paragraph, or a poster. As we will see, it is much harder for a teacher to fill the absence of the other five modeling actions.

MODELING IN ONE TEXTBOOK SERIES

Each of the five actions represents opportunities to model, and textbooks offer students certain opportunities more than others. I will elaborate on each and report my analysis of how well they are represented in textbooks. (See **fig. 1**.)



Identifying Essential Variables

When students identify essential variables in a situation, for example, they are deciding what information matters to a given task and also what does not matter. Teachers undertake this same process in "adult life" all the time. Let's say that you are debating whether renting or buying a home is more cost effective. What information would matter? Interest rates, rent prices, and home prices would all matter, of course. The temperature outside and the number of letters in the street address are much less consequential, of course. And the consequence of information such as the median income of the neighborhood's residents or the median age of their homes is harder to determine. This process is *modeling*—starting with a context and a question alone and asking, "What information is necessary, and what information is unnecessary?"

In my analysis, these two textbooks most often gave students all the necessary information. In only seven of the eighty-three tasks that I analyzed did textbooks withhold some crucial information or ask students to decide what information would be necessary to answer the question.

Formulating Models

In the example of renting or buying a home, once the essential variables have been identified, you formulate models from those variables. Those models might look like tables illustrating the total cost of ownership of the home and the cumulative amount paid in rent over time. Alternately, the model might look like a graph illustrating where those two functions intersect. They might look like equations that will help you locate that intersection point more precisely than will the graph.

It is important that students both see the need for these models and understand them, which is not to say that students need to discover them without teacher support. It is enough that students should experience how disorganized numbers become without a table to organize them, how opaque those numbers become without a graph to visualize them, and how hard it is to make predictions outside our existing data without an algebraic equation.

In 76 percent of the tasks that I analyzed, either the textbook gave the models in the text of the problem, including equations and tables and graphs in the task before students experienced a need for them, or the tasks referred students back to sample problems that featured worked examples of those same models. In my analysis, only twenty of the eighty-three analyzed textbook tasks withheld the model, asking students to derive the graph, equation, or table themselves.

Without doing that hard work, students might come to regard mathematics as a series of formulas to memorize rather than as a series of models that make sense.

Performing Operations and Interpreting Results

Once you have formulated a model, you then perform operations using those models. In our housing example, you know how many years you intend to live in a home, so you calculate the "total cost of ownership" for both options. A calculator is often essential to this step. Then you take the numbers the calculator produces, which are unitless and divorced from any kind of context (the calculator does not know that it is helping decide on housing), and then you interpret the results of those operations. Those numbers represent money. Their units are dollars. And you should choose the option with the lowest number, not the highest.

It may not surprise readers to find out that textbooks give students ample opportunities to practice these modeling skills. In particular, 86 percent of tasks labeled "modeling" asked students to perform an operation—graphing functions, evaluating numbers in an equation, or filling in rows in a table, for several examples. Eighty-one percent of tasks required students to attach units to their answers or interpret them in some other way.

The comprehensive focus of textbooks on these two skills is at odds with their diminishing importance to people who do mathematical modeling for a living. Companies fight to hire people who can thoughtfully generate models for a changing world (the first two modeling actions), but those models are often programmed into computers that automatically handle the second two modeling actions (performing operations and interpreting their results). Or, as Einstein put it nearly a century ago, "The formulation of the problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill" (Einstein and Infeld 1938, p. 92).

Validating Conclusions

When it comes to validating the conclusions of those results, the mathematician George Box was insightful when he said, "All models are wrong, but some are useful" (Box and Draper 1987, p. 424). The mathematical models that we use to describe the world make assumptions that simply will not bear out with 100 percent accuracy. We assume that a runner is modeled by a constant rate equation, not because that is correct but because it is useful. We assume that a file cabinet is modeled by a rectangular prism when we want to figure out how much paper we can store inside it, but the prism is smooth and its walls are infinitesimally thin. None of these assumptions is correct, but they are useful.

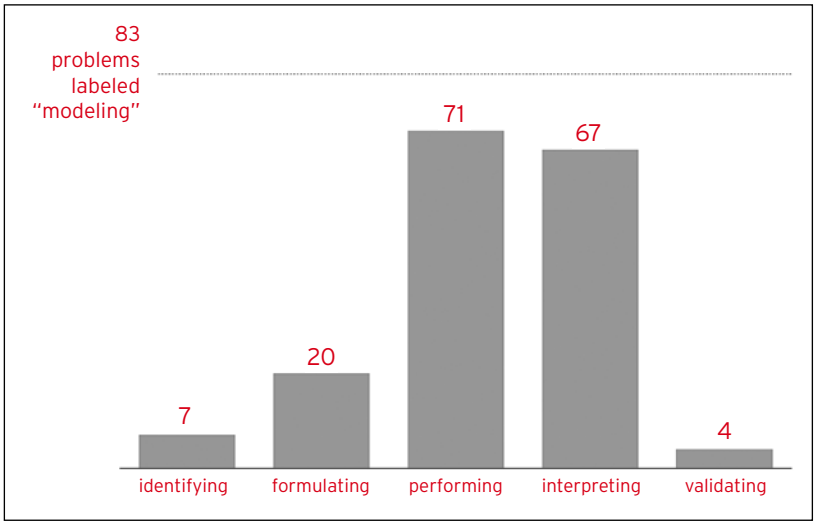


Fig. 1 Modeling opportunities in one textbook series represent different actions.

Sometimes we validate the conclusions of these decisions by performing an internal error check, asking ourselves, "Does this make sense? Have I forgotten anything?" and correcting our models. But sometimes we validate our models when we watch how well they predict—when we find out the runner's actual time for the race or when we try to stuff all this paper we bought into the cabinet. When meteorologists predict the weather, they get to watch the next day to see if their models for weather were valid. The same goes for stock analysts, flight controllers, home buyers—and the list goes on.

The crucial point is that all these models will be wrong. Only in mathematical textbooks do we insist that mathematical models have a zero percent margin of error. We make that warranty when we construct answer keys that match our students' calculated answers without exception. We do students a disservice when we tell them that anything short of 100 percent accuracy means that they have made a mistake. Modeling with zero percent error does not occur in the world of professional modeling or the world of modeling as described in the CCSSM.

Through my analysis, I identified only four tasks out of eighty-three in which the textbook invited students to consider the difference between the conclusions of their mathematical models and real-world results. All four of those tasks were presented in the probability cluster. On the context of probability, these textbooks admitted to students that our mathematical models are not a perfect reflection of the world and that students should verify them experimentally.

RECOMMENDATIONS FOR TEACHERS

The promise of mathematical modeling as written about in the CCSSM and as experienced by professionals has not been realized in these two textbooks, in spite of their claims. These textbooks are strongest where textbooks have traditionally been strongest—in asking students to perform operations and then to attach some cursory meaning to the results of those operations.

So what can we do? The graph in **figure 1** shows the opportunity for teachers. We should step into those gaps and provide our students with more, and more diverse, opportunities to model. Textbooks well emphasize the modeling actions of "performing operations" and "interpreting results"; we teachers need to emphasize the other three modeling actions. I will offer here some suggestions for identifying essential variables in a situation, formulating models from those variables, and validating the conclusions of those results.

If you want to provide your students opportunities to identify essential variables, ask them the question, "What information is necessary here?" whenever you can. It is a question that adults ask themselves intuitively in their grown-up lives all the time. Whenever we encounter a nontrivial challenge, we take our question and then start informing it with data—for instance, "Which grocery line should I get into?" What information matters here? What information matters most? Ask your students, "Should I buy a year-long parking pass or just pay for each day?" Again, the question cannot be answered without more information.

One task in the analyzed textbooks that adequately addressed this modeling action asked students to come up with a way to compare athletes from the same sport. For instance, compare two point guards in basketball. Ask your students, "What information matters here?" Points scored? Definitely. Turnovers and rebounds? Probably. Shoe size and hair color? Likely not, but deciding on essential information requires the joint action of deciding what information is inessential. Would students think that information such as player age, past injuries, and salary is essential or not?

Feel free to stop the conversation there. Do not feel obliged to move through the other modeling actions. It is enough that you have asked and that your students have answered the very grown-up question, "What information do I need here?"

When it comes to formulating models from those variables, we must not think that students need to discover all these complicated mathematical models, although some of them are discoverable. With our point guards, students can ask themselves, "Now that I have identified points, rebounds, and turnovers as my essential variables, what do I do with them to answer my question?" Perhaps the model is as simple as this equation:

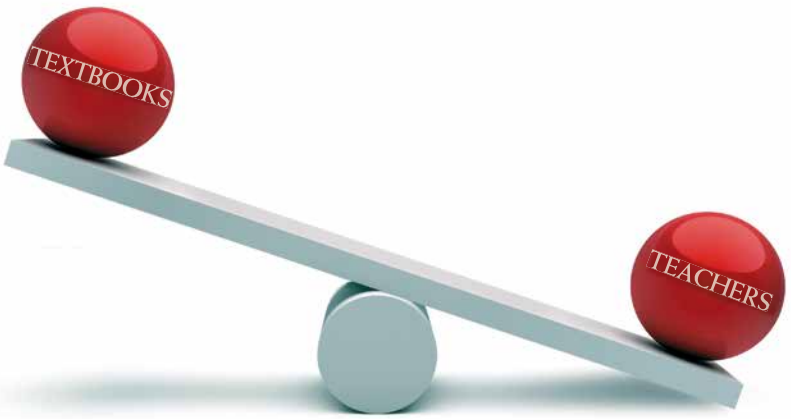
$$\text{player quality} = \text{points} + \text{rebounds} + \text{turnovers}$$

Students may see that this will be a poor model for point guard quality because although points and rebounds are an indicator of player quality, turnovers are bad, and yet they are all added to the model. Rebounds are also good but perhaps not as good as points, so we might revise our model like this:

$$\text{player quality} = \text{points} + 0.5 \cdot \text{rebounds} - \text{turnovers}$$

Other models are formulated less intuitively, and students will need our help constructing them. My precondition for talking about new models is that students should see the "need" for them (Harel 2013), which often occurs when students watch their old models fail them. Proportional models were easy for my students, but a race with a head start required a linear model. Then linear models became easy, but the path of a basketball through the air was modeled very poorly by a linear graph. Once students see their old models fail, we have the opportunity to talk about new models. If students do not see their old models fail, then mathematics becomes a very long tour through all these models that their teachers cherish for no reason that the students can easily discern.

After students have performed operations on a model and interpreted the results of those operations, they will have a prediction. For example, their model may predict that a water tank will empty in eight minutes; their model may predict that the fuel in a car will last only until the next gas station; their model may predict that they will need to buy fourteen pizzas to feed everybody at the party. At this point, students may reasonably



wonder, "Is my prediction correct? How well has mathematical modeling served me here?" Here we should validate students' conclusions by showing the outcome of the event. Show a video of how long it actually took the water tank to fill up. Show the gas gauge at the next gas station. Show a picture of the empty pizza boxes at the end of the party. The world will rarely cooperate completely with our mathematical models; there will be a few leftover slices. But seeing some documentation of their difference—with a photograph, video, personal testimony, or a live demonstration—will help students understand the true of power of mathematical modeling, one of the most powerful tools humans have for understanding their world, although not omnipotent.

Let's worry less about performing operations and interpreting results. Not because they are not important but because the gravity of our textbooks already pulls in their direction. Instead, let's help students see that the world will rarely fully validate the conclusions drawn from their models, that some uncertainty is to be expected, that mathematics is smooth and frictionless, whereas the world is rough and full of surprises. We do that by letting them see the result of their modeling rather than simply telling them those results.

FILLING IN THE GAPS

Not every mathematics task can or should engage students in all five modeling actions. It is possible to practice these actions in isolation from one another also. But this analysis suggests that textbooks offer students limited opportunities to model, tending to complete the first two actions for students (identifying variables and formulating models) while ignoring the last action (validating conclusions). Teachers must make up the difference, offering opportunities for students to engage in those five actions and also to implement tasks that occasionally require all five in concert. (See, for example, Three Act Tasks [threeacts.mrmeyer.com] and 101 Questions [101qs.com]).

I began this article by asking, "What does the CCSSM call 'modeling'?" and "Do textbooks do that definition justice or not?" The answer to the first question is rather clear. The CCSSM authors, working in territory mapped by Box and other applied mathematicians, have provided a precise definition of a complicated and muscular mental act. The answer to the second question is far less certain than I had hoped. My conclusion is not that textbook publishers are diligent or lazy, smart or dumb. My conclusion is that the two textbooks I examined, and arguably numerous others, offer certain modeling opportunities for students and teachers, whereas other modeling opportunities simply are not available for students to learn and teachers to teach.

We must work now to offer students those missed opportunities to model—the kind that lured me into mathematics and kept me in mathematics education, the kind valued by their science classes and valued by employers in satisfying jobs. We can work by asking these different questions in our mathematics classes and making ourselves comfortable with answers from students that will be unpredictable and interesting. We can also work by asking textbook publishers to focus their enormous resources in the direction of interesting and complete mathematical modeling tasks. The CCSSM Publishers Criteria are not binding on any publisher, but our textbook procurement budgets are. Our voice has always been the strongest lever we have for extracting the textbooks we need from publishers. It is time we use it.

REFERENCES

Box, George E. P., and Norman R. Draper. 1987. *Empirical Model-Building and Response Surfaces*. Hoboken, NJ: Wiley.

Carter, John, Gilbert Cuevas, Roger Day, and Carol Malloy. 2013a. *Algebra 1*. Columbus, OH: McGraw-Hill.

———. 2013b. *Geometry*. Columbus, OH: McGraw-Hill.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Einstein, Albert, and Leopold Infeld. 1938. *The Evolution of Physics*. New York: Simon and Schuster.

Harel, Guershon. 2013. "Intellectual Need." In *Vital Directions for Mathematics Education Research*, edited by K. Leatham, pp. 119–51. New York: Springer.

National Science Teachers Association (NSTA). 2012. *Next Generation Science Standards*. http://www.nextgenscience.org/next-generation-science-standards.

Polikoff, Morgan S. 2014. "How Well Aligned Are Textbooks to the Common Core Standards in Mathematics?" Paper presented at the Annual Conference of the Association for Education Finance and Policy, San Antonio, Texas.

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