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12. RME THEORY AND MATHEMATICS TEACHER EDUCATION

The theme of this chapter concerns the question of how mathematics teacher education can prepare prospective teachers for mathematics education that is in line with the domain-specific instruction theory for realistic mathematics education (RME) – which aims at helping students to construct or reinvent mathematics. Before answering this question, a reinvention approach that may avoid the problems that are inherent in more conventional approaches to mathematics education is elucidated. Next, the RME approach is elaborated in terms of the instructional design heuristics, guided reinvention, didactical phenomenology, and emergent modelling. The main question of this chapter is addressed by subsequently investigating (a) what it takes to enable this form of mathematics education in the classroom, (b) how this translates to teacher competencies that are required, and (c) how these competencies may be fostered in mathematics teacher education.

INTRODUCTION

The goal of this chapter is to investigate the implications of the domain-specific instruction theory for realistic mathematics education – RME theory for short – for mathematics teacher education. The general point of departure of RME from conventional approaches is that students should be given the opportunity to reinvent mathematics. According to Freudenthal, who was the founding father of what we now call realistic mathematics education, mathematics has to be (re)invented. This singular point about student (re)invention or construction forms the basis for this chapter. I will therefore spend a part of the chapter justifying this position and try to show the shortcomings of mathematics education that is based on a view of learning as making connections between what you know and what you do not yet know. I will, then, illuminate how RME addresses those issues. It needs to be acknowledged that enacting an RME approach is quite different from theorising about it and that the RME approach is very demanding for teachers. This is why I also elaborate on the requirements that need to be fulfilled in order to bring about instruction that is consistent with RME principles.

The chapter has the following structure. I start by discussing the common view of learning as making connections between what one knows, and what one needs to learn. I do so by asking the question, ‘What makes mathematics so difficult?’ Next I argue that the alternative notion of learning as constructing or (re)inventing offers a better chance of helping students learn mathematics. Then, I describe the RME approach as an example of a domain-specific instruction theory that tries to give

directions for how to guide students in such a process. I do this by elaborating RME in terms of three instructional design heuristics: (1) guided reinvention, (2) didactical phenomenology, and (3) emergent modelling. Finally, I turn to the question, what does it take to effectuate the intended form of mathematics education in the classroom? In this respect, I discuss the need for a ‘reinvention route’, the willingness of students to (re)invent, and the competency of the teacher to guide the reinvention process. These three requirements will be translated into teacher competencies, which are used for a final discussion of how teacher education may provide for such competencies.

LEARNING AS MAKING CONNECTIONS

The difficulty of learning mathematics is often explained by referring to the gap between the student’s personal knowledge and the abstract formal mathematical knowledge that needs to be acquired. From a constructivist perspective, however, it may be argued that the problem is not simply in the gap that has to be bridged. The problem is that, for the student, there is nothing at the other side of the bridge. The gap-metaphor presupposes an objective body of knowledge that exists independently of some agent. According to constructivism, knowledge is constructed by someone, and cannot be separated from the constructing individual. Thus for those who have not yet constructed the more sophisticated mathematical knowledge that has to be learned, this more sophisticated mathematical knowledge, literally, does not exist.

Nevertheless, the gap-metaphor seems to be rather generally treated as plausible. This may be explained by the fact that we, as adult mathematics educators, *perceive* our own abstract mathematical knowledge as an independent body of knowledge. We experience mathematical objects such as ‘tens’, ‘ones’, and ‘hundreds’, for instance, or ‘linear functions’, to mention another example, as object-like entities that can be pointed to and spoken about. This experience relates not only to our individual mathematical sophistication, but also to our experience of being able to talk and reason about these ‘objects’ unproblematically while interacting with others. As a consequence, we may assume that we can talk and reason about these ‘objects’ with students as well. Likewise, teachers and textbook authors may take their own abstract mathematical knowledge for an external body of knowledge, which can be communicated to students. The difference between the abstract knowledge of the teachers and the experiential knowledge of the students, however, constitutes a serious source of miscommunication – as ample research shows.

Identifying a problem, however, is not the same as solving it. Constructivism in itself does not give an answer to how to teach mathematics. To elaborate this point, we may turn to Cobb’s (1994) discussion of the notion of ‘constructivist pedagogy’. He starts by observing that constructivism is often reduced to the mantra-like slogan that ‘students construct their own knowledge’. A mantra that, he argues, is not only erroneously treated as a fact that is beyond justification, but also as a direct instructional recommendation. Concerning the latter, the common line

of reasoning is that, since the students necessarily construct their own knowledge, the teacher's role is limited to that of facilitating students' investigations and explorations. Cobb (1994, p. 4), however, argues,

On alternative reading, the constructivist maxim about learning can be taken to imply that students construct their ways of knowing in even the most authoritarian of instructional situations.

In other words, the assumption that students construct their own knowledge cannot be directly translated into an instructional recommendation. This does not mean that constructivism cannot play a role in developing an instructional approach for mathematics education, but the critical issue is not whether students are constructing, but *what* and *how* they are constructing. Thus, taking a constructivist perspective implies that one has to consider the question, what it is that we want the students to construct, and how we want them to construct it.

LEARNING AS CONSTRUCTING OR INVENTING

Many years ago, Freudenthal (1971, 1973) addressed the theme of what mathematics is, or what we want it to be for our students, from a different angle. He took his point of departure in the notion of 'mathematics as a human activity'. Being a mathematician himself, he characterizes mathematics as the activity of mathematicians that involves solving problems, looking for problems, and mathematizing subject matter. The latter may concern mathematizing mathematical matter, or mathematizing subject matter from reality, in which mathematizing stands for organizing subject matter from a mathematical point of view. In his view, the main activity of mathematicians is that of mathematizing. The final stage of this activity is formalizing by way of axiomatizing. The result of this activity, he goes on to say, is taken as a starting point in traditional mathematics instruction. He calls this an anti-didactical inversion, for the endpoint of the work of generations of mathematicians is taken as the starting point for the instruction of students. As an alternative, he advocates for giving students the opportunity to do what mathematicians do. Instead of presenting mathematics as a ready-made product, he goes on to say, the primary goal of mathematics education should be to engage students in mathematics as an activity. Then, similar to the way in which the mathematical activity of mathematicians has resulted in mathematics as we know it, the activity of students should result in the construction of such mathematics. In this scenario, the students have to be supported in inventing mathematics. In this respect, Freudenthal (1973) speaks of *guided reinvention*. Guidance by teachers and textbooks is not only needed to ensure that the mathematics that the students invent corresponds with conventional mathematics, but also to substantially curtail the invention process. Students cannot simply reinvent the mathematics that took the brightest mathematicians eons to develop. Teachers need to help students along, while trying to make sure that the students experience their learning as a process of 'inventing' mathematics.

Over time, Freudenthal's ideas have been elaborated in the so-called domain-specific instruction theory for realistic mathematics education (RME), which I elucidate in the following paragraphs. I start with an example, which concerns the constitution and flexible use of a framework of number relations up to 20.

Flexible Arithmetic Up to 20 as an Example

In the above, we marked the significant difference between the abstract knowledge of teachers and the experiential knowledge of students. Following Freudenthal (1991), we might even speak of different realities. He defines reality as, 'what (...) common sense experiences as real' (Freudenthal, 1991, p. 17). He argues that what is common sense for a layman is different from what is common sense for a mathematician. A similar distinction applies to teachers and students. We may illustrate this with the following example.

At a certain age, young children do not understand the question: How much is $4+4$? Even though they may at this stage very well understand that 4 apples and 4 apples make 8 apples. The explanation for this apparent paradox is that, for those children, a number do not yet have an independent meaning in and of itself. For these children, numbers are tied to countable objects, as in 'four apples', 'four marbles', or 'four ice creams'. 'Four' is more like an adjective than a noun for them. At a higher level, '4' will be associated with various number relations, such as: $4 = 2+2 = 3+1 = 5-1 = \frac{8}{2}$, and so on. At this higher level, numbers have become mathematical objects that derive their meaning from a network of number relations (Van Hiele, 1973). We might in fact speak of the construal of a new mathematical reality in which numbers are experienced as mathematical objects.

When elementary-school teachers talk about numbers, they may very well be speaking about mathematical objects, which are not part of the students' experiential realities. The result is that teachers and students in fact speak different languages – without being aware of it. Teachers talk about numbers as mathematical objects that exist within a network of numerical relations. They may, for instance, explain that '7+6 equals 13 because $7+3 = 10$, $6 = 3+3$, and $10+3$ equals 13'. The students, however, who have not yet construed the necessary network of numerical relations and think of numbers as adjectives, cannot follow this line of reasoning. The result of this miscommunication will be that the students will have to revert to copying and memorizing.

According to Van Hiele (ibid), we may avoid this problem by helping students to construct a network of number relations, within which numbers have become mathematical objects. The goal will be that the students will come to have the experience of directly perceiving numerical relationships as they interpret and solve arithmetical problem situations. That is to say that they will be able to flexibly solve tasks such as '7+6' by using numerical relationships that will be readily available for them, such as ' $7+3 = 10$, $6 = 3+3$, and $10+3$ ', or ' $6+6 = 12$, $6+1 = 7$, and $12+1 = 13$ ', or ' $7+7 = 14$, $6-1 = 7$ and $14-1 = 13$ '. Although we may perceive these solutions, as applying strategies – such as 'filling up ten', and so on,

this does not have to be the case for the students. The instructional intent is that the students will be guided by their familiarity with number relations, and do not have to think of strategies.

This analysis suggests that the first step in a reinvention approach would be to involve the students in activities of structuring quantities in a wide variety of situations, to make them aware of the number relations involved, and to help them construct number relations by generalizing over the various situations. The question that arises here is which relations should be focused upon? To find an answer to this question, one may start by looking at research on students' informal solution strategies. Research shows that students frequently develop strategies that make use of the doubles, and five and ten as points of reference (Gravemeijer, 1994). The spontaneous use of five and ten as reference points can be traced back to the creation of finger patterns (Van der Berg & Van Eerde, 1985; Treffers, 1991). Assuming that these patterns themselves may emerge as curtailments of counting on the fingers by one, instruction may start by working with finger patterns. Students may be asked, for instance, to show 'eight' in different ways, and the teacher may draw the attention to the number relations involved. Eight is shown as '5+3', or as '4+4', whereas it may also be construed as '10-2'. As a next step in a reinvention route, we may want students to use this knowledge to derive new number relations. As a means of support, students might use a so-called *arithmetic rack* (Treffers, 1991, see Figure 1), which is designed to support numerical reasoning in which five, ten, and doubles are used as points of reference.

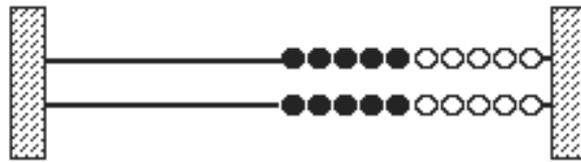


Figure 1. Arithmetic rack.

This device consists of two parallel rods each containing ten beads. The first five beads on the left of each rod are red, and the second five beads are white. Students use the rack by moving all beads to the right and then creating various configurations by sliding beads to the left. For example, if a student wants to show eight, he or she may move five beads on the top rod and three on the bottom, or he/she may move four beads on each rod. These ways of acting with the arithmetic rack may facilitate the use of the relations that come to the fore in finger patterns.

As a caveat, it should be noted that there is a danger of superficial learning in this set up. In fact, the only gain of the first example may be that, instead of being told that $4+4$ equals 8, the students now 'come to see' that $4+4$ equals 8. To avoid this, the instructional activities have to surpass the level of merely structuring sets of objects and reading of answers. Instead, we would want students be able to

reason that $4+4=8$, on the basis of some counting procedure, for instance. An important issue here is that the number patterns that the students construct come to signify curtailments of the procedure for quantifying sets of objects by counting individual objects. In this manner, visual patterns, such as finger patterns, come to embody the results of counting, to use Steffe, Cobb, and von Glasersfeld's (1988) terms. This implies that the students have to construe procedures for establishing sums or differences – such as counting on, and counting back – as extensions of the counting procedure that they use for quantifying sets of objects. The importance hereof is shown by research of Gray and Tall (1994), who observe that students, who have come to see the first and the latter as two unrelated procedures, do not use 'derived facts' strategies.

Similar risks are attached to the arithmetic rack. Here too, the intent is not that students will use the arithmetic rack configurations to read off number relations. Instead, students are expected to use the arithmetic rack as a means of scaffolding. To be able to use the arithmetic rack in this manner, students already have to have developed five-, ten-, and doubles-referenced number relations. To find the sum of 6 and 7, for instance, the students may then use their knowledge that $6=5+1$ and $7=5+2$ to visualize 6 and 7 on the rack as 5 red and 1 white on the top rod and 5 red and 2 white on the bottom rod, then to subsequently take the two fives together and reason, that $6+7 = 5+1+5+2 = 10+3 = 13$. Or they may realize that $7=6+1$, and $6+6=12$, and relate this to of 6 on the top rod and 6 at the bottom and reason $7+6 = 12+1 = 13$, while pointing to the rack.

As our example shows, the design of an instructional sequence that may give rise to a reinvention process is rather complicated. We would not expect teachers to design such instructional sequences themselves. This is a far more demanding task than what is usually taken on in lesson studies. Fortunately, however, that will not be necessary. Since the early 1970s, researchers/instructional designers at the Freudenthal Institute and elsewhere have worked on developing instructional sequences that would fit Freudenthal's conception of guided reinvention. An important aspect of this work was the explication of the rationales behind each of the instructional sequences. Such a rationale, or *local instruction theory*, consists of a theory about a possible learning process for a given topic, and the means of supporting that process. The theory is called local in that it is tailored to a given topic, such as addition of fractions, multiplication of decimals, or data analysis. Each local instruction theory offers a description of, and rationale for, the envisioned learning path as it relates to a set of instructional activities for a specific topic.

The local instruction theories developed at the Freudenthal Institute have been taken as a basis for the construal of a more general instructional theory. By generalizing over those local instruction theories, Treffers (1987) deduced, what he called, a framework for a domain-specific instruction theory for realistic mathematics education. Later, RME theory was recast in terms of three design heuristics: guided reinvention, didactical phenomenology, and emergent modelling (Gravemeijer, 1989), which I will discuss in the following section.

RME THEORY

Guided Reinvention as an Instructional Design Heuristic

Guided reinvention not only describes the overall approach of RME, but it can also be seen as an instructional design heuristic. Taken as a heuristic for design, the reinvention principle suggests the instructional designer look at the history of mathematics to see how certain mathematical practices developed over time. The designer is advised to especially look for potential conceptual barriers, dead ends, and breakthroughs. These may be taken into account when designing a potential reinvention route. As a second guideline the reinvention principle suggests investigating whether students' informal interpretations and solutions might 'anticipate' more formal mathematical practices. If so, students' initially informal reasoning can be used as a starting point for the reinvention process. In summary, the designer may take both the history of mathematics and students' informal interpretations as sources of inspiration for delineating a tentative, potential route along which reinvention might evolve.

As a special point of attention we may note that reinvention has both an individual and a collective aspect, it is the interaction between students in particular that function as a catalyst. The designer needs to develop instructional activities that are bound to give rise to a variety of student responses. What is aimed for is a variety in responses that to some extent mirrors the reinvention route. When some students come up with more advanced forms of reasoning than others, teachers can exploit these differences. They can try to frame the mathematical issue that underlies those differences as a topic for discussion. In orchestrating such a discussion, they can then foster the reinvention process. Without such differences, the teacher will not have a basis for organizing a productive classroom discussion, and will have to ask leading questions to solicit the preferred responses.

In relation to this, we may note that reinvention is intimately tied to the activity of mathematizing, more to vertical mathematization than to horizontal mathematization (which is more tied to mathematizing problem situations). In relation to this, we may distinguish between mathematical interest and pragmatic interest. One of the points of departure of RME is that contextual problems should not only be experientially real, the problems also have to be realistic. It has to be plausible for the students that someone wants to know the answer, thus the context has to offer a reason for wanting to know the answer. In this manner, students are to be motivated to solve contextual problems for pragmatic reasons. Vertical mathematizing, however, requires them to be interested in the mathematical aspects, for mathematics sake. This mathematical interest may not come naturally but has to be cultivated by the teacher by asking questions such as: What is the general principle here? Why does this work? Does it always work? Can we describe it in a more precise manner? We may assume that the teacher can foster the students' mathematical interest by making mathematical questions a topic of conversation, and showing a genuine interest in the students' mathematical reasoning.

Didactical Phenomenology as an Instructional Design Heuristic

The second RME design heuristic concerns the didactical phenomenological analysis, or *didactical phenomenology* for short (Freudenthal, 1983). Here the word ‘phenomenological’ refers to a phenomenology of mathematics. In this phenomenology, the focus is on how mathematical ‘thought-things’ (which may be concepts, procedures, or tools) organize – as Freudenthal (1983) puts it – certain phenomena. Knowing how certain phenomena are organized by the *thought thing* under consideration, one can envision how a task setting in which students are to mathematize those phenomena may create the need for them to develop the intended *thought thing*. In this manner, problem situations may be identified, which may be used as starting points for a reinvention process. Note that such starting-point-situations may also be used to explore the students’ informal strategies. To find the phenomena that may constitute starting-point-situations, we may look at applications of the concept, procedure or tool under consideration. Assuming that mathematics has emerged as a result of solving practical problems, we may presume that the present-day applications encompass the phenomena, which originally had to be organized. Consequently the designer is advised to analyze present-day applications in order to find starting points for a reinvention route. Note, however, that as the students progress further in mathematics, applications may concern mathematics itself. Essential for valuable starting points is that they are experientially real for the students, that those concern situations, in which the students know how to act and reason sensibly.

Emergent, Modeling as an Instructional Design Heuristic

The third RME design heuristic is called *emergent modelling* (Gravemeijer, 1999). This design heuristic takes its point of departure in the activity of modelling. Modelling in this conception is an activity that students may employ when solving a contextual problem. Such a modelling activity might involve making drawings, diagrams, or tables, or it could involve developing informal notations or using conventional mathematical notations. The conjecture is that acting with the models will help the students to reinvent the more formal mathematics that is aimed for. Initially, the models come to the fore as context-specific models. The models refer to concrete or paradigmatic situations, which are experientially real for the students. Initial models should allow for informal strategies that correspond with situated solution strategies at the level of the situation of the contextual problem. Then, while the students gather more experience with similar problems, their attention may shift towards the mathematical relations and strategies. This helps them to further develop those mathematical relations, which enables them to use the model in a different manner: The model becomes more important as a base for reasoning about these mathematical relations than as a way of representing a contextual problem. In this manner, the model starts to become a means of support for more formal mathematics. Or more precisely: A *model of* informal

mathematical activity develops into a *model for* more formal mathematical reasoning.

Underlying this transformation is a gradual shift in level of activity, from 'referential level' to a 'general level' (Gravemeijer, Cobb, Bowers, & Whitenack, 2000). At the referential level, the model derives its meaning for the students from its reference to activity in the task setting. With help of the teacher, the attention is shifted towards the mathematical relations involved. At the general level, the model starts to derive meaning from these mathematical relations, and starts to become a model for more formal mathematical reasoning. Finally the students may reach the level of more formal mathematical activity, when a new piece of mathematical reality is formed, and mathematical reasoning is no longer dependent on the support of a model. Note that although we may speak of a model that is first constituted as a '*model of*' that gradually changes into a '*model for*', the students actually will be working with a series of sub-models, which may take the form of inscriptions or tools. From the perspective of the researcher/designer however, the series of sub-models constitute an overarching model. It is this overarching model that co-evolves with some new mathematical reality. The emergent, modelling design heuristic asks of the designer to explicate this new mathematical reality, i.e., the framework of mathematical relations and the mathematical objects that constitute this mathematical reality. This explication is not only important for instructional design, it can also inform teachers about what mathematical relations to focus on in classroom discussions.

An issue of concern is that even though the designer intends for the use of inscriptions or tools to be experienced as bottom-up by the students, this will not necessarily be the case. The teacher will therefore have to try to monitor, what new sub-models signify for the students. Here we may use the term 'imagery' to refer to the question of whether acting with the tools evokes an image of earlier activities, on the basis of which the students can make sense of the new sub-models.

In addition, measures may be taken to ensure that new (sub-) models come to the fore as a natural extension of earlier activities. The teacher may, for instance, introduce a new way of symbolising in an informal off-hand manner, and wait and see if the students will appropriate this way of notating. A decisive criterion here is whether the students adopt and adapt the new form of symbolising in a flexible manner. Another way may be to try to create the need for a new tool by problematizing the current state of affairs, and asking the students for their solutions. Then, after that the solutions offered by the students have been discussed, the teacher may present the next tool as the solution that was chosen by someone in the given context. Even if the students do not come up with the next tool, the preceding discussion would at least have provided a basis on which the students could conclude that the teacher's proposed solution was sensible under the given conditions.

RME Design Heuristics Clarified with a Local Instruction Theory for Flexible Arithmetic up to 20

We may briefly return to our example of flexible arithmetic up to 20, to further elucidate the aforementioned design heuristics. The reinvention principle suggests to not just teach some ready-made strategies, such as ‘filling up ten’, or ‘using doubles’. Instead, the designers will ask themselves how flexible mental computation might emerge. The analysis of informal solution procedures showed that students may develop a framework of number relations that offers the building blocks for flexible mental computation (see also Greeno, 1991). In addition, we observed that ‘counting on’ and ‘counting back’ have their roots in counting as a means for establishing quantities.

For the didactical phenomenological analysis, we may refer to Freudenthal (1983), who observes that numbers organize the phenomenon of quantity, while addition organizes phenomena such as combining two sets – as in 5 cars and 3 cars or 5 marbles and 3 marbles. There are, however, he adds, other situations, where addition is not plainly recognizable as the union of two sets. Take for instance, John has 5 marbles, and Pete has 3 more. How many does Pete have? Here, the students must consider the imaginary set of Pete as split into two sets, and reason from there. He goes on to say that there are also spatial or temporal phenomena where one cannot speak of a union of two unstructured sets. With spatial or temporal phenomena such as adding 5 steps (of stairs) and 3 steps, 5 days and 3 days, or 5 kilometres and 3 kilometres, counting is used to organize magnitudes, in which measuring the magnitude is articulated by the natural multiples of a unit. Continuous phenomena are made discrete by a one-to-one mapping of the successive intervals on a sequence of points that follow each other in space or time, in a process that in turn suggests a counting process. In line with this sequential character, the results of additions of magnitudes are obtained by counting on. In relation to this, Freudenthal (ibid, p. 99) points to the close relation between cardinal and ordinal numbers: ‘ $5+3$ is defined cardinally, but from olden times it has been calculated ordinarily’. The result of $5+3$ is obtained by starting with the mental 5, and counting on, 6, 7, 8. At the same time, it shows that counting strategies, such as counting on and counting back, rely on integrating the cardinal aspect of number (quantity) and the ordinal aspect of number (position/rank). Most addition and subtraction problems concern quantities, while the solution procedures consist of moving up and down the number sequence. From this we may conclude that it is important that the students connect the first and the latter.

For emergent modelling, we may focus on the role of the arithmetic rack, as it was elaborated in a research project in Nashville, Tennessee (Gravemeijer, Cobb, Bowers, & Whitenack, 2000). The arithmetic rack is designed with potential useful number relations in mind. In this manner, the students may use the rack as a means of scaffolding basic number relations, and as a basis for developing more elaborate frameworks of number relations. On a more practical level, the designer will first have to look for situations that can be modelled with the arithmetic rack. Here one may make use of instructional activities that are designed by Van den Brink

(1989), which involve the double-decker bus scenario. Initial tasks then concern different ways in which a given number of passengers could sit on the two decks of a double-decker bus. Follow-up activities involve situations in which some passengers get on and others get off the bus. Next the arithmetic rack can be introduced as a means of showing the number of passengers on each deck, and as a means of keeping track of the number of people getting on or off the bus. In this way, the rack can initially function as a model of (the changes in) the number of passengers on the two decks. An important step in the modelling process will then be to ask the students to develop ways of notating their reasoning with the rack so that they can communicate it to others. Subsequent activities involve developing and negotiating symbolizations, the key criterion being that other children in the class could understand how the task had been solved. For example, the ways of reasoning with rack about $7+8$ might be symbolized as shown in Figure 2.

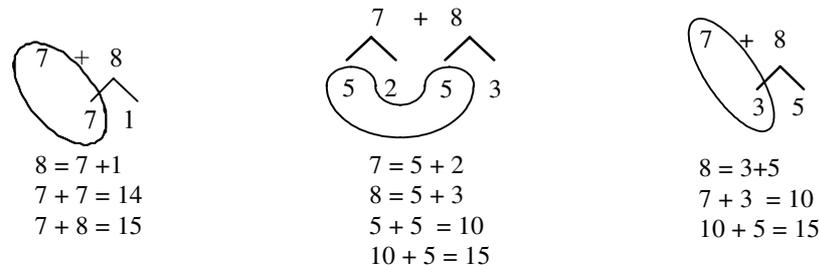


Figure 2. Ways of visualizing arithmetic rack solution strategies.

By then, drawings of the way of reasoning with the rack will be functioning as models for more formal mathematical reasoning. Finally the student may start using number sentences without any auxiliary drawings.

A Local Instruction Theory on Data Analysis as an Example

To clarify the emergent modelling design heuristic a bit further, I will briefly discuss another example of a local instruction theory that is taken from a teaching experiment on data analysis carried out by Cobb, Gravemeijer, McClain and Konold in a 7th-grade (12 year-olds) classroom in Nashville (see also Gravemeijer & Cobb, 2006). The general goal of this local instruction theory is that the students come to view data sets as distributions, of which one can discern characteristics that are relevant when resolving issues concerning the situation where the measurements have been taken. The starting points for the instructional activities are realistic problems that provide a reason for analysing data. Having a reason is essential in our view, for this rationale guides the very process of data analysis. In conventional statistics courses, statistical measures like mean, mode, median, spread, quartiles, (relative) frequency, regression, and correlation are taught as a set of independent definitions. These statistical measures, however, are characteristics

of distributions. So, for these measures to have meaning, students have to have a notion of distributions as objects that can have certain characteristics. Likewise, conventional representations like histogram and box plot come to the fore as means to characterize distributions. We reasoned therefore, that instead of teaching statistical measures and representations as such, one should focus on helping students in developing the notion of distribution as an object.

The notion ‘distribution’ is closely tied to the graph with which we visualize a distribution. Distribution then can be thought of in terms of shape, density, and position. A way to think about such a graph is as a density function. This offers a way into a qualitative understanding of distribution. In such a conceptualization, the height of a point on the graph signifies the density of data points around that value. From a didactical phenomenological point of view, we may speak of the graph as a means for organizing density. Density in turn can be seen as means for organizing collections of data points in a space of possible data values. From the same phenomenological perspective, data points in a dot plot may be thought of as a way of getting a handle on a set of data. With such an analysis, we already have a rough outline of a series of sub models. In relation to this, we may describe the overarching model as ‘a graphical representation of the shape of a distribution’.

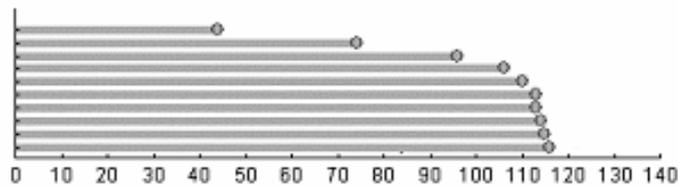


Figure 3. Value-bar graph.

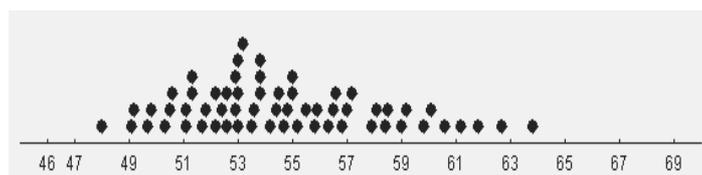


Figure 4. Dot plot.

The most common graph of a distribution is the graph of a density function we discussed earlier. However, the graph to start the sequence with would have to be a graph that would most closely match an intuitive visualization of a measure for the students. This in our view is a scale line. Especially measures of a linear type, like ‘length’, and ‘time’ are often represented by scale lines in primary school. These

considerations let to the decision to start with a graph that consists of value bars, where each value bar signifying a single measure (Figure 3). Within a magnitude-value-bar graph shape, the distribution of the data values is visible in the way the endpoints of the value bars are distributed in regard to the axis. In relation to this, we can speak of a *graphical representation of the distribution* as a *model* of a set of measures. Next the students may come to see the dot plot as a more condensed form of a line plot that leaves out the value bars and only keeps the end points (Figure 4). Within a dot plot, the density of the data points in a given region translates itself in the way the dots are stacked. Consequently, the height of the stacked dots at a given position can be interpreted as a measure for the density at that position. In this sense, the visual shape of the dot plot can be seen as a qualitative precursor to the graph of a density function. This aspect can be further developed by having the students' structure data into four equal groups, when resolving issues concerning the situations where the measurements have been taken. They may come to see the distance between two vertical bars that mark a quartile in a four-equal-groups display as indicating how much the data are 'bunched up'. Moreover, the median may start to function as an indicator of 'where the hill is' in a uni-modal distribution. Finally, the students are expected to begin to treat distributions as entities. In this regard, we may describe the four-equal groups displayed as a *graphical representation of the distribution* that started to function as a *model* of a model for reasoning about distributions.

WHAT DOES IT TAKE TO ENACT RME?

RME Theory and Local Instruction Theories

After having depicted RME theory, I now move to the question, what are the implications of this theory for mathematics teacher education? I try to derive those implications from considering the question, what does it take to bring about instruction that is in tune with RME? I discern three requirements. As a first requirement, I argue that one has to have a sound idea about what the intended reinvention process may look like for a given topic. That is to say one has to have a plan for a possible reinvention route. A second requirement is that the students have to be willing to invent, which is less self-evident than it may sound. The third requirement concerns the capability of the teacher to support the intended reinvention process. I elaborate those three requirements in the following paragraphs.

The Need for a Planned Reinvention Route

In my view, designing reinvention processes is a very complicated task that surpasses the scope of what may be expected of teachers. I want to argue therefore that teachers should be offered a more general framework that enables them to design instructional activities on a day-to-day basis. Such a framework may be offered by a so-called local instruction theory, which consists of a theory about a

possible learning process for a given topic, and the means of supporting that process. A valuable feature of RME theory is that it is (being) developed by way of generalizing over exemplary instructional sequences, or local instruction theories. A consequence of this is that RME theory comes with a set of local instruction theories, which are consistent with RME. Each local instruction theory offers a description of, and rationale for, the envisioned learning path as it relates to a set of instructional activities for a specific topic (such as ‘addition and subtraction up to 20’, ‘area’, ‘fractions’, and so forth). Those local instruction theories can function as frameworks of reference for teachers. Here we may refer to Simon’s (1995) notion of a ‘hypothetical learning trajectory’. Simon argues that a teacher who wants to build on the students’ thinking and activity, and at the same time work towards given learning goals has to consider what mental activities the students might engage in as they participate in the instructional activities, he or she is considering. A decisive criterion of choice then would be how those mental activities relate to the chosen learning goal. In relation to this, Simon (*ibid*) speaks of designing a hypothetical learning trajectory. He emphasizes the hypothetical character of these learning trajectories; the teachers are to analyse the reactions of the students in light of the stipulated learning trajectory to find out in how far the actual learning trajectory corresponds with what was envisioned. Based on this information the teacher has to construe new or adapted instructional activities in connection with a revised learning trajectory.

Local instruction theories can function as frameworks of reference in this process. The relation between the local instruction theory and the hypothetical learning trajectories can be elucidated with a travel metaphor (Simon, 1995). In terms of a travel metaphor, the local instruction theory offers a ‘travel plan’, which the teacher has to transpose into an actual ‘journey’ with his or her students. The idea is that the teachers will use their insight in the local instruction theory to choose instructional activities, and to design hypothetical learning trajectories for their own students. Here the teachers will orient themselves to the actual thinking and reasoning of their students. Consequently, they may look for forms of assessment that are tailored to revealing student thinking and reasoning. In line with this idea a RME approach to assessment has been developed that aims at creating opportunities for students to show what they know and can do. Instead of, as is often the case with a test, showing what students are not (yet) able to. This kind of assessment is also called ‘didactical assessment’ (Van den Heuvel-Panhuizen & Becker, 2003) in that it tries to find footholds for instruction that may follow the test.

I close this section by noting, that the issue of how to document local instruction theories is not really resolved yet. The traditional form of textbooks and teacher guides will not be adequate. For, key in the kind of instruction that I discuss here is that it is responsive to the input of the students, and adaptable to the beliefs and concerns of the teachers. A possible solution may lay in an elaboration of the ‘model-of/model-for’ shift that can be taken as the backbone of an RME instructional sequence. As noted before, that model can take on different manifestations during the realization of an instructional sequence. Those sub-

models can be described as a chain of signification (Whitson, 1997), which details not only the manifestations and progression of the model, but also the evolving taken-as-shared meaning and purposes of the classroom community. The local instruction theory then can be described as an anticipated chain of signification, by describing the anticipated tools that will be used and the imagery, mathematical activity and mathematical practices that correspond to them (Gravemeijer, 2004).

The Willingness of Students to (Re)Invent

In the above, I extensively discussed one of the basic prerequisites for reinvention, the plan for a reinvention route. Another prerequisite is that the students are willing to invent. This seems a rather trivial point. I have to take into account, however, that students are often familiar with a classroom culture in which the classroom social norms (Cobb & Yackel, 1996) are that the teacher has the right answers, that the students are expected to follow given procedures, and that correct answers are more important than one's own reasoning. In this type of classrooms teachers usually ask questions of which they already know the answer. Apart from being used to this situation, students have learned what to expect and what is expected from them. In relation to this, Brousseau (1988) speaks of an implicit 'didactical contract'. Significant, however, is that the students have learned this by experience, not because the teacher told them so. This may be illustrated by research of Elbers (1988), who asked students in kindergarten, 'What is heavier, red or blue?' The students gave different answers, but what was striking that they gave answers at all. The explanation for this is that they already held very specific expectations for their role. For them, it was quite normal for the teacher to ask questions they could not answer. They had learned that they had to give an answer, any answer, to enable the teacher to proceed. Later on they might learn in retrospect why that question was asked.

In contrast to the traditional classroom culture, reinvention asks for an inquiry-oriented classroom culture. The classroom has to work as a learning community. To make this happen, the students have to adopt classroom social norms such as the obligation to explain and justify their solutions. They have to be expected to try and understand other students' reasoning, and to ask questions if they do not understand, and challenge arguments they do not agree with. In addition to those social norms, Cobb and Yackel (1996) discern socio-mathematical norms. Where the classroom social norms are, in a sense content free – they could yield for any topic – socio-mathematical norms relate to what mathematics is. These socio-mathematical norms encompass, what counts as a mathematical problem, and what counts as a mathematical solution. Important also is, what counts as a more sophisticated solution, for this relates to vertical mathematizing. One has to have a norm for expecting what is mathematically more advanced to be able to advance in a reinvention process. In this respect, I may argue that socio-mathematical norms provide the basis for the intellectual autonomy of the students, as it enables them to decide for themselves on mathematical progress.

In addition to appropriating inquiry-based norms, students also have to be willing to invest effort in solving mathematical problems, discussing solutions, and discussing the underlying ideas. The students' willingness to participate in learning activities is the outcome of a reciprocal process that depends on both individual attributes and contextual features such as classroom climate and instructional support. Students may engage in learning activities for different reasons. The attitude of students in a mathematics classroom can be broadly divided in two categories, *ego orientation* and *task orientation* (Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990). On the one hand, ego orientation implies that the student is very conscious of the way he or she might be perceived by others. Ego-oriented students are afraid to fail, or to look stupid in the eyes of their fellow students, or the teacher. As a consequence, they may choose not to even try to solve a given problem, in order to avoid embarrassment. Task orientation, on the other hand, implies that the student's concern is with the task itself, and on finding ways of solving that task. Research shows that task orientation and ego orientation can be influenced by teachers.

Cobb, Yackel and Wood (1989) report on a study on a socio-constructivist classroom, where task orientation was fostered. Part of their approach was to change the classroom culture from one of competition, where students would compare themselves with each other, and with the criteria set by the teacher, into a classroom culture, where students would measure success by comparing their results with their own results earlier. One may think of the latter perspective as one similar to that of an amateur painter or amateur musician. An amateur musician would not think of comparing him- or herself with others; there would always be many people performing better. Instead an amateur musician would be pleased if he or she would master a piece, which he/she could not play some time ago. A similar situation is possible in a mathematics classroom, where the goal for the students would be personal growth. Here, experiencing an 'Aha-Erlebnis', will function as an incentive. In such a classroom, students might even protest at being given 'the solution', for that would deprive them from the satisfaction of figuring out things for themselves. Such a process of figuring out may very well have the character of collaborative work, where the students see themselves as community that works towards shared understanding. In fact, the aforementioned research of Cobb et al. (1989) shows that a classroom culture that emphasizes the exchange of ideas, and the development of mathematical understanding as a collaborative endeavor, fosters the task orientation of the students. It may be noted, however, that teachers need to strike a balance between creating freedom for the students to figure things out by themselves and offering support. Research of Turner, Midgley, Meyer, Gheen, Anderman, Kang, and Patrick (2002) shows that too much freedom or challenge can have a negative effect on student's feeling of self-confidence, when the students may have too thin a knowledge base to build on.

Teacher Competencies for Guiding the Reinvention Process

The aforementioned requirements already imply a central role for the teacher. The role and competencies of the teacher constitute a third requirement for reinvention. One may structure those roles and competencies in three categories, one that concerns the planning and design of instructional activities and hypothetical learning trajectories, one that concerns the classroom culture, and one that concerns the orchestration of the collective reinvention process.

A central competency in the category *planning* is that of designing, evaluating, and revising instructional activities and hypothetical learning trajectories that both fit the current state of affairs in the classroom and a given local instruction theory. In connection with this the teacher has to be able to identify experientially real starting points, a competency that may be supported by the ability to design and use didactical assessment tasks. Another competency concerns the identification of mathematical objects and mathematical relations at which the instructional activities are aimed.

In the category *classroom culture*, of course, the competency to establish and maintain the intended classroom social norms and social-mathematical norms comes to the fore as a central competency. In addition to this I may mention the ability to cultivate mathematical interest, and to foster task orientation over ego orientation. An additional competency will be that of the ability to instill in the students an orientation towards a perspective of personal growth.

The first two categories are complemented by competency to orchestrate the *collective reinvention process*. The teacher has to be able to see his or her plans through, and to grasp and assess the students' thinking and reasoning. He or she has to be able to introduce a new way of symbolizing in an informal off-hand manner, or create the need for a new tool, and assess whether the new tool signifies earlier activities, and if the students adopt and adapt symbolizations or tools in a flexible manner. An additional competency concerns the ability to see what differences in mathematical understanding underlie the variation in student responses, to frame mathematical issues as topics for discussion on the basis of this insight, and to orchestrate productive whole-class discussions on those mathematical issues.

HOW TO FOSTER RME TEACHER COMPETENCIES IN TEACHER EDUCATION

The above list of teacher's roles and competencies sets the agenda for teacher education. It will be impossible to give a detailed account of how mathematics teacher education can help students in developing those competencies. A golden rule, however, will be to teach what you preach. That is to say that the prospective teacher has to be educated in a way that mirrors the way they are expected to teach by themselves. One of the main themes will be to enable the prospective teachers to work with local RME instructional theories. In particular, a course such as this may be designed to have the students experience classroom norms, and reflect on how those norms are established and maintained. They may also experience, and

reflect upon, the role of imagery and history in the use of tools and inscriptions. And in a similar manner, task orientation, and personal growth may be addressed in a course that aims at hypothetical learning trajectories and local instruction theories. For now, I will limit myself to a brief elaboration of such a course.

Designing hypothetical learning trajectories on the basis of (externally developed) local instruction theories and resource materials differs significantly from following scripted textbook teacher guides. Mathematics teacher education therefore has to educate prospective teachers in working with local instruction theories. An overriding goal of such a preparation is to help them to develop an autonomous attitude, which allows them to take the liberty to interpret and adapt local instruction theories. More specifically, prospective teachers learn to construe, evaluate, and revise hypothetical learning trajectories on the basis of local instruction theories. In this respect it is fortunate that most local instruction theories are the product of design research (Gravemeijer & Cobb, 2006).

The methodological norm in design research is that the learning process of the researchers/designers should justify what they claim to have learned. In relation to this, one speaks of trackability (Smaling, 1992). Research reports should offer outsiders the opportunity to retrace the learning process that the designer/researcher went through. Or as Freudenthal (1991, p. 16) put it (who speaks of ‘developmental research’ instead of ‘design research’):

Developmental research means: ‘experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become their own experience.

Ideally, prospective teachers should be given the opportunity to experience such learning processes by some form of reinvention. Solving sequence related problems, anticipating solutions of primary or secondary school students, analysing student work and teaching episodes and such could be the constituents of such a reinvention process – where analysing student work and teaching episodes may take the form of multimedia video case studies (Dolk, Hertog, & Gravemeijer, 2002). In addition, prospective teachers should be made familiar with the overall educational philosophy that underlies the local instruction theories that they may want to use. Otherwise, it will be difficult to come to grips with these instruction theories. This overall philosophy is an integral part of the justifications of local instruction theories. For, a local instruction theory is not just a theory of how to teach a given topic; it is a theory about how to teach that topic within the framework of a certain philosophy of mathematics education.

It may not be realistic to expect the prospective teachers to develop a detailed understanding of all local instruction theories he or she might need. Instead, prospective teachers might learn to make sense of new theories on their own accord. In relation to this it will be important that the prospective teacher comes to grips with key principles that hold for all local instruction theories within a given framework – which concern mathematics as an activity, guided reinvention, didactical phenomenology, and emergent modelling. This goal may be

accomplished by having prospective teachers analyse, experiment with, and reflect on, some exemplary local instruction theories. In such activities, they might gain insight in the task of developing hypothetical learning trajectories. Moreover, the key principles of RME may emerge in reflective activities under the guidance of the mathematics teacher educator. Apart from this, it can be argued that prospective teachers should be made familiar with a whole set of local instruction theories that covers the curriculum. Insight in the key principles underlying these local instruction theories may help to come to grips with the essence of these local theories.

In conclusion we may note that if we want to create mathematics classrooms within which students construct their own knowledge, mathematics teacher education must play a central role. RME can be seen as an exemplary elaboration of such an approach to mathematics education. The work from RME has produced a wide variety of instructional activities, instructional sequences, and (local) instructional theories. These, however, are merely resources; in the end, it is the teacher who enacts realistic mathematics education in the classroom. And, we will have to acknowledge that this kind of teaching is extremely demanding, which in turn possess a challenge to mathematics teacher education, both for prospective and practising teachers.

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