KOENO GRAVEMEIJER and MICHIEL DOORMAN

# CONTEXT PROBLEMS IN REALISTIC MATHEMATICS EDUCATION: A CALCULUS COURSE AS AN EXAMPLE

ABSTRACT. This article discusses the role of context problems, as they are used in the Dutch approach that is known as realistic mathematics education (RME). In RME, context problems are intended for supporting a reinvention process that enables students to come to grips with formal mathematics. This approach is primarily described from an instructional-design perspective. The instructional designer tries to construe a route by which the conventional mathematics can be reinvented. Such a reinvention route will be paved with context problems that offer the students opportunities for progressive mathematizing. Context problems are defined as problems of which the problem situation is experientially real to the student. An RME design for a calculus course is taken as an example, to illustrate that the theory based on the design heuristic using context problems and modeling, which was developed for primary school mathematics, also fits an advanced topic such as calculus. Special attention is given to the RME heuristic that refer to the role models can play in a shift from a *model of* situated activity to a *model for* mathematical reasoning. In light of this model-of/model-for shift, it is argued that discrete functions and their graphs play a key role as an intermediary between the context problems that have to be solved and the formal calculus that is developed.

## 1. INTRODUCTION

The role of context problems used to be limited to the applications that would be addressed at the end of a learning sequence – as a kind of add on. Nowadays, context problems have a more central role. They are endorsed because of today's emphasis on the usefulness of what is learned, and because of their presumed motivational power. Context problems play a more encompassing role in the Dutch approach that is known as realistic mathematics education (RME). In RME context problems play a role from the start onwards. Here they are defined as problems of which the problem situation is experientially real to the student. Under this definition, a pure mathematical problem can be a context problem too. Provided that the mathematics involved offers a context, that is to say, is experientially real for the student.

In RME, the point of departure is that context problems can function as anchoring points for the reinvention of mathematics by the students

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themselves. Moreover, guided reinvention offers a way out of the generally perceived dilemma of how to bridge the gap between informal knowledge and formal mathematics. This issue is at the heart of this article: *How can we help students to come to grips with formal mathematics?*

We will take a calculus course as an example, and show that in the reinvention approach, the role of context problems and of symbolizing and modeling are tightly interwoven. Actually, we build upon the work that has been done on symbolizing and modeling in primary-school mathematics (Streefland, 1985; Treffers, 1991; Gravemeijer, 1994, 1999). We try to show that the framework that has been developed for primary school can also be used for such an advanced topic as calculus.

We start by following Tall's critique on a formal approach to calculus teaching, by explicating the problems of instruction based on formal logical analysis. Next we discuss some alternatives before moving on to an elaboration of the RME approach.

The RME calculus sequence is inspired by the history of mathematics. We will describe some elements of the history of calculus from the findings at the Merton College in the 14th century until Galileo that are interesting from an instructional design point of view. We argue that discrete functions and their graphs played a key role as an intermediary between the context problems that had to be solved and the formal calculus that was being developed. We will finish with a discussion of the RME approach of creating the opportunity to let formal mathematics emerge, instead of trying to bridge a gap between formal and informal knowledge.

## 2. Traditional Calculus Instruction and Some Alternatives

### *Traditional set up*

According to Tall (1991), mathematicians tend to make a typical error when they design an instructional sequence for calculus. The general approach of a mathematician is to try to simplify a complex mathematical topic, by breaking it up in smaller parts, that can be ordered in a sequence that is logical from a mathematical point of view. 'From the expert's viewpoint the components may be seen as part of a whole. But the student may see the pieces as they are presented, in isolation, like separate pieces of a jigsaw puzzle for which no total picture is available.' Tall (1991, p. 17). It may be even worse, if the student does not realize that there *is* a big picture. The student may imagine every piece as an isolated picture, which will severely hinder a synthesis. The result may be that the student constructs an image of each individual piece, without ever succeeding in bringing all pieces together in one whole.

As an example, Tall describes a possible sequence for differentiation. He presents the following line of reasoning. To be able to understand the derivative $f'(x)$, one has to have the concept of a limit at one's disposal. For, one has to take the limit of the difference quotient $(f(x+h)-f(x))/h$ where $h$ tends to zero. Thus the concept of a limit has to precede the derivative. Furthermore, one might decide that it is easier to take the limit in the case where $x$ is fixed. The next step then would be to let $x$ vary, to introduce the idea of a derivative in this manner.

For the student, however, the introduction of the limit concept suddenly appears for no reason, with all the cognitive problems this may bring. The next big problem is in the shift from a limit with a fixed x to a varying $x$, since taking a limit in one point is substantially different from perceiving $f'(x)$ as a function of which the values describe the gradient of a graph of $f(x)$.

*Alternative approaches*

A sensible alternative, according to Tall, would be to look for situations that can function as informal starting points, from which cognitive growth is possible. In this context, Tall (cited in Bishop et al., 1996) argues for more emphasis on visualizing mathematical concepts and more enactive experiences in mathematics education. The students should first experience a qualitative, global, introduction of a mathematical concept. This qualitative introduction then should create the need for a more formal description of the concepts involved. 'Graphic calculus' is such an alternative approach developed by Tall (1985, 1986).

What is characteristic in Tall's approach is the dynamical aspect. The dominant perspective is that of a running variable. Here the graph plays a central role: the graph is used to see how the dependent variable (e.g. y, or $f(x)$) changes when the independent value ($x$) changes at a constant rate. What is looked at primarily, are the changes in the dependent variable and the rate of these changes, which can be described in terms of increase, decrease, and gradient. This helps the students develop an intuitive feel for the derivative. However, the notion of the derivative as a measure for the rate of change stays rather implicit, since the primary focus is on the gradient of the graph.

Tall (cited in Bishop et al., 1996, p. 314) observes that the problem for a calculus course is in the transition from meaningful discussions based on visual imagery to formal mathematical reasoning. Students interpret a definition that is based on visual imagery as a description, as a model of the picture, instead of a mathematical definition that can be used for formal reasoning.

Maybe the visual exploration stands too much on its own. In Tall's proposal, the relationship between graphs and real-life phenomena is restricted to a short introductory phase; for the main part, the functions and graphs are contained in a mathematical world.

### *Linking authentic experience and a mathematical symbol system*

In contrast to Tall, Kaput (1994a), emphasizes the relationship between mathematical symbol systems like graphs and everyday reality.

The problem, in his view, is the gap between the island of formal mathematics and the mainland of real human experience. He elucidates this gap with the difference between mathematical functions that are defined by algebraic formulas, and empirical functions that describe everyday-life phenomena. To underscore his point, he quotes Thomas Tucker's rhetorical question: 'Are all functions encountered in real life given by closed algebraic formulas? Are any?' (cited in Kaput, 1994a, 384). He observes that most educational software available does not address this problem. This, he argues, also holds for computer programs that link functions and graphs. For, in terms of the island metaphor, both algebraic functions and graphs of algebraic functions belong to the island of formal mathematics.

To attack the island problem, Kaput seeks situations where the students can maximally exploit their own authentic experience to investigate, and to come to grips with these formal representations. He tries to create such a situation with software he is developing under the name of Math Cars. The power of the device is in the internal linkage between the various display systems. In this way, the everyday experience of motion in a vehicle can be linked to the formal graph representations. This linkage then offers the students the opportunity to test the conjectures they develop about the graphic representations.

We may note that Kaput takes the ready-made symbol system as point of departure, which is consistent with his concept of a mathematical system that is distinct from our everyday-life experience. Others, however, try to help students develop or reinvent this symbol system themselves.

### *Inventing graphing*

DiSessa, Hammer, Shern, and Kolpakowski (1991), who describe an instruction period where the students invented graphing, for instance, report such an approach. Albeit, not as a result of ample instructional planning. As a matter of fact, the invention process was more or less incidental. The students had been programming simulations of real-life motions with a

Logo-like turtle that left a trail of dots across the screen. Next, the students were asked to come up with a paper-and-pencil way to represent the motion story of one of the simulations they worked on. The solutions of the students, that were to some extent inspired by the dot-track of the computer simulation, formed the starting point for a series of discussions and activities, in which a graph-type representation of this motion story emerged.

While acknowledging that the guidance by the teacher may have influenced the invention process more than the notion of invention suggests (DiSessa et al., 1991), we can argue that, if it would be possible to have the students invent distance-time and speed-time graphs by themselves, the dichotomy between formal mathematics and authentic experience Kaput presupposes, would not arise. For the mathematical ways of symbolizing would emerge in a natural way in the students' activities, and the accompanying formal mathematics would be experienced as an extension of their own authentic experience.

Following Meira (1995), we can take this line of thinking one step further, by acknowledging a dialectical relation between notations-in-use and mathematical sense making. According to this dynamic point of view, it is in the process of symbolizing that symbolizations emerge and develop their meaning. In this process, notational systems shape the very activities from which they emerge, while at the same time, the activities shape the meanings that emerge.

*Discussion*

Looking at the alternatives to traditional calculus instruction presented above, we may discern the following characteristics. The overall goal is to design an insightful instructional sequence. Furthermore, the students should be given the opportunity to ground their understanding in their own informal knowledge. Finally, the instruction should be based on the students' own contributions to the teaching-learning process. We may discern three different orientations within these alternatives:

- Helping the students to develop qualitative (generic) notions that can function as a basis for their understanding;
- Creating a learning environment where the students can come to grips with the basic ideas by developing and testing hypotheses;
- Fostering the (re)invention of calculus.

A difference between the first two and the third is that the first two try to help the students bridge the gap between their informal knowledge and the formal mathematics, where the third tries to *transcend this dichotomy* by aiming at a process in which the formal mathematics emerges from the mathematical activity of the students. This is also the objective of RME, where instructional design is aimed at creating optimal opportunities for the emergence of formal mathematical knowledge.

## 3. Guided reinvention and progressive mathematization

The underlying thread of all the alternative approaches is the belief that learning mathematics should have the characteristics of cognitive growth, and not of a process of stacking pieces of knowledge. This perspective is consistent with a more general view that the way in which mankind developed mathematical knowledge, is also the way in which individuals should acquire mathematical knowledge. A view that is, for instance, advanced by Polya (1963), and Freudenthal (1973, 1991).

Freudenthal's point of departure is in his critique of traditional mathematics education. He fiercely opposes what he calls an anti-didactical inversion (Freudenthal, 1973), where the end results of the work of mathematicians are taken as the starting points for mathematics education. As an alternative he advocates that mathematics education should take its point of departure primarily in *mathematics as an activity*, and not in mathematics as a ready-made-system (Freudenthal, 1971, 1973, 1991). With this adage, he has laid the foundation for RME. For him the core mathematical activity is 'mathematizing', which stands for organizing from a mathematical perspective. Freudenthal sees this activity of the students as a way to reinvent mathematics.

Note that the students are not expected to reinvent everything by themselves. In relation to this, Freudenthal (1991) speaks of *guided reinvention*; for him, the emphasis is on the character of the learning process rather than on invention as such. The idea is to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible. The latter implies that certain social norms (Yackel and Cobb, 1996) have to be in place. For instance, norms like: you do not learn mathematics by guessing what the teacher has in mind, but by figuring things out for yourself.

According to Freudenthal, mathematizing may involve both mathematizing everyday-life subject matter and mathematizing mathematical subject matter (Freudenthal, 1971). He does not see a fundamental difference between the two activities. Therefore, education might start with mathematizing everyday-life subject matter. Reinvention, however, demands that the students mathematize their own mathematical activity as well. In relation to this, Treffers (1987) discerns horizontal and vertical mathematization. Horizontal mathematization refers to the process of describing a context problem in mathematical terms – to be able to solve it with mathematical means. Vertical mathematization refers to mathematizing one's own mathematical activity. Through vertical mathematization, the student reaches a higher level of mathematics. It is in the process of progressive mathematization – which comprises both the horizontal and vertical component – that the students construct (new) mathematics.

Freudenthal (1971, p. 417) expresses this as 'the operational matter on one level becomes a subject matter on the next level'. Although Freudenthal has micro levels in mind, a connection can be made with Sfard's (1991) more macroscopic account of mathematical development based on historical analyses. She observes that the history of mathematics can be characterized as an ongoing process of reification in which processes are reinterpreted as objects. One of the examples she gives is that of functions. They firstly appear as calculational prescriptions. This process takes so much attention that the character of the operations does not get much attention. Gradually, however, distinctions are made, and various sorts of functions are discerned. More and more, functions are treated as objects, with certain characteristics. Sfard's analysis suggests that students will have to go through the same process; students will only be in the position to grasp the notion of a function as an object, if they have ample experience with functions as procedures.

### *An instructional design perspective*

In a reinvention approach, context problems play a key role. Well-chosen context problems offer opportunities for the students to develop informal, highly context-specific solution strategies. These informal solution procedures then, may function as foothold inventions, or as catalysts for curtailment, formalization or generalization. In short, in RME, context problems are the basis for progressive mathematization. The instructional designer tries to construe a set of context problems that can lead to a series of processes of horizontal and vertical mathematization that together result in the reinvention of the mathematics that one is aiming for. Basically, the guiding question for the designers is: *How could I have invented this?* Here the designer will take into account his/her own knowledge and learning experience. Moreover, the designer can look at the history of mathematics as a source of inspiration, and at informal solution strategies of students who are solving applied problems for which they do not know the standard solution procedures yet (see Streefland, 1990; and Gravemijer, 1994 for examples).

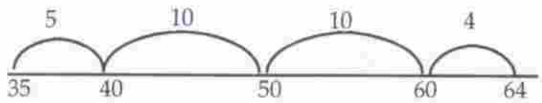

*Figure 1.* Solving 35 + . . . = 64 on the empty number line

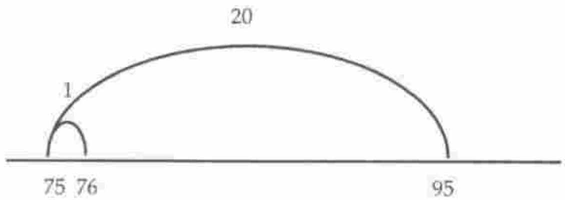

*Figure 2.* Modeling 95 − 19 = 95 − 20 + 1.

Research on the design of primary-school RME sequences has shown, that the concept of *emergent models* can function as a powerful design heuristic (Gravemeijer, 1999). Here, the point of departure is in situation-specific solution methods, which are then modeled. First context problems are selected that offer the students the opportunity to develop these situation-specific methods. Then, if they do, these methods are modeled. In this sense, the models emerge from the activity of the students. Even if the models are not actually invented by the students, great care is taken to approximate student invention as closely as possible by choosing models that link up with the learning history of the students. Another criterion is in the potential of the models to support vertical mathematizing. The idea is to look for models that can be generalized and formalized to develop into entities of their own, which as such can become models for mathematical reasoning.

As an example we may take an instructional sequence in which a ruler comes to the fore as a *model of* iterating measurement units, and develops into a *model for* reasoning about mental computation strategies with numbers up to 100 (Stephan, 1998; Gravemeijer, 1999). In this sequence, the students measure various lengths by iterating some basic unit of measurement, and a larger measurement unit consisting of ten basic units. This measuring with 'tens' and 'ones' is modeled with a ruler of 100 units, made of units of ten and one. Then the activity of measuring is extended to incrementing, decrementing and comparing lengths. These situations give rise to counting strategies that are represented by arcs on a schematized ruler or as an 'empty number line' (Whitney, 1988; Treffers, 1991) (see Figure 1).

Eventually, the symbolizations on the empty number line representation will be used to explain and justify strategies like solving 95–19 by subtracting 20 and adding one (see Figure 2). In this manner, the number line functions as a model for mathematical reasoning.

The shift from model/of to model/for concurs with a shift in the way the student thinks about the model, from models that derive their meaning from the modeled context situation, to thinking about mathematical relations. In the latter phase, thinking about number relations will dominate the use of the number line. In relation to this, we can discern different types of activity (Gravemeijer, 1994; Gravemeijer, Cobb, Bowers and Whitenack, in press):

(1) activity in the task setting (measuring with units of ten and one)
(2) referential activity (interpreting positions on the ruler as signifying results of iterating a measurement unit)
(3) general activity (using the ruler/number line to reason about computation methods)
(4) formal mathematical reasoning (reasoning with number relations within the mathematical reality of a framework of number relations).

Note that, the term 'model' must be understood in a holistic sense. It is not just the inscription[1], but everything that comes with it that constitutes the model in RME. Furthermore, the same model may encompass a cascade of inscriptions (Lehrer and Romberg, 1996); e.g. from regular ruler to empty number line. The label 'emergent' emphasizes the continuity in this process. This label is also used to refer to the fact that the model emerges from the activity of the students. In addition, the mathematics that one is aiming for (e.g. the development of a framework of number relations), emerges in the subsequent process.

We may note that the goal is not only to help the students elaborate their informal understanding and informal solution strategies in such a manner that they can develop more formal mathematical insights and strategies. The objective is also to preserve the connection between the mathematical concepts and that which these concepts describe. The students' final understanding of the formal mathematics should stay connected with, or as Freudenthal would say, should be 'rooted in', their understanding of these experientially real, everyday-life phenomena.

In relation to this, we want to emphasize that we see modeling and symbolizing as an integral part of an organizing activity that aims at coming to grips with a problem situation. In other words, the development of the model and of its constituting symbolizations/inscriptions go hand in hand with the development of the mathematical conceptualization of the problem situation. On one hand, the symbolization derives its meaning from the situation that it describes. At the other hand, the way the problem situation is perceived is highly influenced by the symbolization 'through which' the situation is seen. In this sense, we agree with Meira (1995, p. 270) who advocates a dynamic activity-oriented view, according to which symbolizations and meaning co-evolve in a dialectic process.

In the following we want to try to cast the design of the RME calculus course in terms of this emergent-models design heuristics. We will start out by sketching a few important moments in the history of calculus. Since, although reinvention does not necessarily imply that the history of mathematics has to be the point of departure, this RME calculus course happens to be an example of how inspiring the history of mathematics can be for mathematics educators. Moreover, the history of calculus offers some insight into what might be constituted as the emergent model in calculus.

## 4. A HISTORICAL SKETCH

### *The emergence of kinematics at Merton College*

In the first half of the 14th century logicians and mathematicians associated with Merton College, Oxford, investigated velocity as a measure of motion. They tried to find a description of the distance traversed by a body moving with a uniform accelerated motion (Clagett, 1959). This problem was not easy because the velocity of the body constantly changes and the concept of motion and change of motion was defined very generally according to Aristotle. Change and motion referred to temperature, size and place (Lindberg, 1992). All these qualities of a body could change. The interpretation of motion as change of place became the central issue studied at Merton College; Clagett describes this as 'the emergence of kinematics at Merton College'.

In those days the scientists at Merton College used the notion of instantaneous velocity and descriptions of the velocity of a moving object. But there was no clear definition of velocity such as: the distance covered divided by the traversal time ($\Delta s/\Delta t$). And surely no definition of instantaneous velocity as a limit of this division ($ds/dt$), because they could only work, in the tradition of Euclid, with proportions of equivalent units.

The most important result that was achieved is 'Merton's rule': When the velocity of an object increases with equal parts in equal time intervals from zero to a velocity $\mathbf{v}$ in a time interval $\mathbf{t}$, then the distance traveled is equal to half the distance traveled by an object that moves with a constant velocity $\mathbf{v}$ in that time interval $\mathbf{t}$. In modern symbols $\mathbf{s(t) = 1/2 \cdot v \cdot t}$.

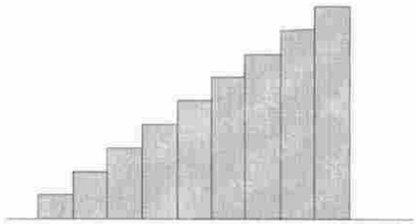

*Figure 3.* Velocity represented by the height of the rectangles.

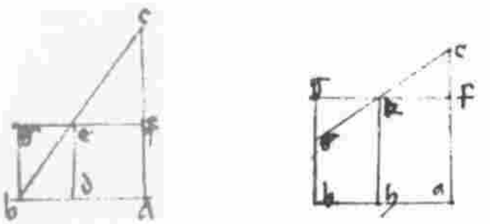

*Figure 4.* Oresme's proof (pictures from a 15th century copy of Oresme's 'De configurationibus qualitum' reprinted from Claget, 1959).

They added the conjecture that the distance covered in the first half of such a motion is one third of the distance covered in the second half. However, this appeared to be hard to prove.

*The graph as a model of velocity-time problems*

It was Nichole Oresme (ca. 1360) who started to draw graphs of situations in order to visualize the problem. He described how geometrical elements like lines or rectangles can be used to represent the value of a variable: the length of a line or the area of the rectangle represents the value of the variable. These elements can be put along a horizontal line that represents time.

When investigating the distance traveled by an object that moves with constantly changing velocity Oresme knew that the velocity increases with equivalent quantities in equivalent time intervals[2]. A graph where the velocity is represented by the height of a rectangle, and the chosen time interval by the width of it, is shown in Figure 3.

Oresme could approximate the distance traveled in each time interval by taking into account the length of the interval and the 'constant' velocity during the interval. The area of the rectangle represents this distance. Oresme called the diagonal top-line of the graph, which results when you choose the time-intervals very small, 'the line of intensity'. He noticed that it is a straight line and that the sum of the areas of all the rectangles is an approximation of the area of the triangle between the line of intensity and the horizontal axis. As a result the distance traveled equals the area of the triangle: $\mathbf{1/2 \cdot v \cdot t}$ (Figure 4).

The power of this graphical representation comes to the fore when you try to prove the hypothesis: the distance covered in the first half of a uniformly accelerated motion is one third of the distance covered in the second half. This can be deduced easily with the areas in the graph.

*Galileo investigating free fall*

The interesting fact of these results is that they dealt with abstract phenomena. Kinematics was an argumentative science, not an experimental science. It was Galileo in the seventeenth century that made the shift to experimental physics. He conjectured that an object in free fall moved with a constantly increasing velocity. He designed experiments with objects moving with a uniform acceleration and applied the Merton rule to a real motion.

Furthermore, Galileo gave an explanation for the quadratic relation between time and distance, using sums and differences. He argued that the ratio between the distance traveled in two equal time intervals is 1:3; if you divide time into four equal intervals it is 1:3:5:7; and it continues as such a ratio determined by a sequence of odd numbers. Galileo knew that these sequences added up to a square and with this property he tested his conjecture. He designed a slide with nails on one side. The distances between the nails were proportional to the odd numbers above. It appeared that a rolling ball needed the same time too pass each consecutive nail.

Galileo's experiments confirmed that the falling distance increased linearly during each time-interval and from this result he concluded that the relation between distance and time must be quadratic. The integral of the linear function was thus found; an actual start with calculus was made. We leave history here, to consider what we can learn from this.

*Looking through the lens of emergent models*

Looking at the history of calculus from a modeling perspective, we see a development of calculus that starts with modeling problems about velocity and distance. Initially these problems are tackled with discrete approximations, inscribed by discrete graphs. Later, similar graphs – initially discrete and later continuous – form the basis for more formal calculus. From a modeling perspective, we could say that graphs of discrete functions come to the fore as *models of* situations, in which velocity and distance vary, while these graphs later develop into *models for* formal mathematical reasoning about calculus. In relation to this we can speak of a process of reification. However, it is not the graph that is reified, it is the activity that is

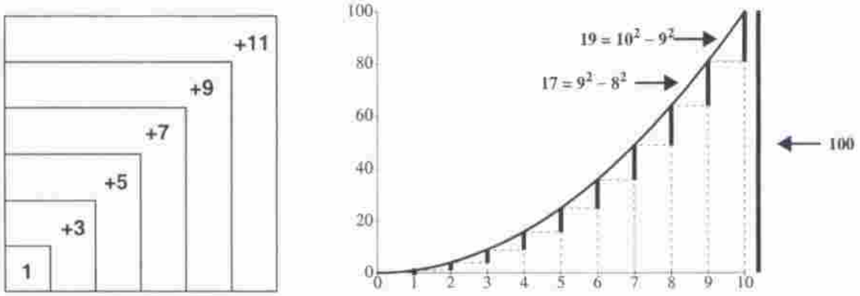


*Figure 5.* Sums and increments.

reified. E.g. the act of summing (differentiating) is reified, and becomes the mathematical object 'integral' ('derivative'). The inscription – the graph – visually supports both the activity and its reification.

We want to add that we do not take this notion of reification too literally here. For many students, the end result will be something in between a process and an object – especially in the case of calculus. To emphasize that the underlying process will be an integral part of the mathematical object that is developed in this manner, Tall (cited in Bishop et al., 1996) uses the term 'procept'. Moreover, it should be acknowledged that the development will not be as linear as our description suggests. Students may shift back and forth between process and object conceptions, depending on the problems they have to solve.

In looking at the history of calculus, Kaput's (1994b) characterization of calculus as 'the mathematics of change' comes to mind. In the process of trying to get a handle on change, the method of approximating a constant changing velocity with the help of discrete functions plays a key role. In a sense, the sum- and difference-series can be seen as a crude predecessor of calculus. This idea can be exploited in instructional design by starting the sequence with investigating discrete patterns. Moreover the mathematics of sum- and difference-series offers the opportunity of reinventing a discrete variant of the main theorem of calculus in an early stage.

Another aspect of the history that seems relevant for instructional design is the embedding in the context of speed and distance. This seems to be a context that is suited for high school students. For, in general, speed is a concept they still grapple with. Moreover the idea of instantaneous velocity seems to be more assessable than a seemingly simple concept such as average speed.

These deliberations are elaborated in a calculus sequence that is developed for students of 16–17 years, in science-oriented high school classes[3]. The unit starts with activities on series. Properties of, and the relations between series, their sums and their increments are investigated by the students (see Figure 5). In this process, the students develop (algebraic) tools that can be used for problems later on. Moreover, it is a first introduction to the relationship between sums, summation symbols, increments and difference symbols. The notion of limit is only used in an intuitive way until the end of the unit.

The relationship between sums and differences is represented with the introduced symbols by: $\Delta n^2 = 2k + 1$ and $\sum(2k + 1) = n^2$. These properties are also visualized with graphs. In general the discrete case of the main theorem is posed: $\Delta S(n) = D(k)$ and $\sum D(k) = S(n) - S(0)$. Students use graphic calculators when dealing with these investigations. From graphs and tables they can deduce that $\Delta 3^n = 2 \cdot 3^k$ and, with the theorem they can prove that $\sum 3^k = 1/2(3^n - 3^0)$.

After this discrete analysis the shift towards modeling of time/distance/velocity is made. Note that a key element of the notion of emergent models is that the models first come to the fore as models of situations that are experientially real for the students. It is in line with this notion that discrete graphs are not introduced as an arbitrary Symbol System, but as models of discrete approximations of a motion. Since the students are already familiar with continuous graphs, an approach to graphing speed that is not familiar to the students, is chosen. The point of departure is in the medieval notion of instantaneous speed. The issue of symbolizing speed is introduced in the context of a narrative about Galileo's work. The students are informed about the definition of instantaneous velocity in Galileo's time in terms of the distance that is covered if the moving object maintains its instantaneous velocity for a given period of time. In this context, the problem is posed of how to visualize the motion of an object that moves with varying speed. While struggling with this problem, the students may come up with the idea of symbolizing instantaneous velocities with bars. If not, this option may be presented to them, after they have been struggling with this problem for a while.

As an aside, it may be noted that there will always be a tension between a bottom-up approach that capitalizes on the inventions of the students and the need, (a) to reach certain given educational goals, and (b) to plan instructional activities in advance. As a consequence, a top-down element is inevitable in instruction. The key consideration for us, however, is that the

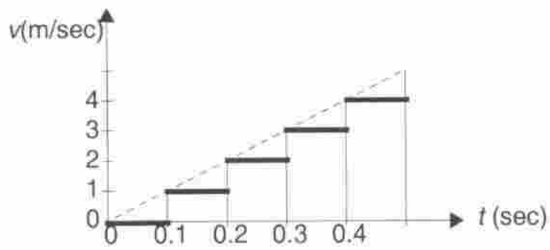


*Figure 6.* A discrete approximation of a constant changing velocity.

students experience these top-down elements as bottom-up: as solutions they could have invented for themselves. For the instructional designer, this boils down to striving to keep the gap between 'where the students are' and what is being introduced as small as possible. Furthermore, the teachers will be able to reduce the gap in interaction with their students.

Returning to the content, we want to stress, that although the idea of using bars (i.e. small rectangles) may be handed to the students, this does not mean that there are no problems left to solve. A central problem is that of coordinating the height and the width of the bars when using them to visualize a discrete approximation of a movement.

Next, the topic of investigation will be the relation between the 'area of the graph', and the total distance covered over a longer period of time. Figuring this out demands of the students that they come to grips with the relationship between the motion, the representation, and the approximation. The whole process, in which the way of modeling motion, and the conceptualization of the motion that is being modeled, co-evolve, can be seen as a form of guided reinvention that can be contrasted with learning some rule which magically equates area with distance.

The students are told the story of Galileo, who presumed that the motion of a free-falling object was with constantly increasing velocity. They are asked to graph the discrete approximation of such a motion (see Figure 6).

Subsequently Galileo's problem is posed: What distance is covered by the object? Here, after solving the discrete approximations, the students are expected to make the connection between the area of the discrete graph, and the area of the triangle that is created by the continuous graph: $s(t) = (t \times 10t)/2 = 5t^2$. This resulting formula reveals the quadratic relation between time and distance that Galileo used to test his hypotheses empirically. With this calculation, the first step is made in a process in which the attention shifts from describing motion, to calculating primitives.

After this introduction to integral calculus, students work on problems that deal with differential calculus. That is, they have to determine velocities from distance-time graphs and formulas. In subsequent activities, the model will begin to function as a model for reasoning about integrating/differentiating arbitrary functions on the one hand, and of standard algebraic functions on the other hand. At the same time, a shift is made from problems cast in terms of everyday-life contexts to a focus on the mathematical concepts and relations. To make such a shift possible for the students, they will have to develop a mathematical framework of reference that enables them to look at these types of problems mathematically (see Gravemeijer, 1999). It is exactly the emergence of such a framework that this sequence tries to foster.

In the envisioned process of progressive mathematization, symbolizing and modeling play a key role. The central model is that of a discrete function, in combination with the inscription of a discrete graph. This model is the basis for both integration and differentiation through sums and differences. Although the discrete function can be seen as the backbone of the model, the visual representation also plays an important role. In line with Latour (cited in Meira, 1995) we would argue that the use of visual representations helps one to focus on the mathematics.

## 6. Discussion, creating mathematical reality

The question we asked at the beginning of this article was: How can we help students to come to grips with formal mathematics? Central in our exposition was the RME approach. This approach distinguishes itself from many other approaches, in that it tries to transcend the dichotomy between informal and formal knowledge, by designing a hypothetical learning trajectory along which the students can reinvent formal mathematics. Ideally, the actual learning trajectory unfolds in such a manner, that the formal mathematics emerges in the mathematical activity of the students. This ideal is connected to Freudenthal's (1991) contention that 'mathematics should start and stay within common sense'. Freudenthal intended this adage to be interpreted dynamically and argued that common sense is not static. He noted, for example, that what is common sense for a mathematician differs significantly from what is common sense for a lay person. In addition, he emphasized that common sense evolves in the course of learning. Thus, in the first phase of the sequence, describing momentary speed in terms of the distance that would be covered in case of constant speed, is a common-sense activity. In a similar manner, a discrete approximation of varying velocity can be seen as a common sense activity. By the end of the sequence, acting in an environment structured in terms of areas and gradients of graphs will have become common sense for the students.

This development can also be taken to exemplify what is meant by context problems in RME. As we indicated earlier, context problems are defined as problem situations that are experientially real to the student. The above example shows that this experiential reality grows with the mathematical development of the student. Freudenthal explicates: 'I prefer to apply the term 'reality' to that which at a certain stage common sense experiences as real' (Freudenthal, 1991, p. 17). This use of the term reality in RME is highly compatible with Greeno's (1991) environmental metaphor. From this we conclude that the overall goal of instructional design is to support the gradual emergence of a taken-as-shared mathematical reality. If the students experience the process of reinventing mathematics as expanding common sense, then they will experience no dichotomy between everyday life experience and mathematics. Both will be part of the same reality.

We may note a reflexive relation between the use of context problems and the development of the student's experiential reality. On the one hand, the context problems are rooted in this reality, on the other hand, solving these context problems helps the students to expand their reality. Notwithstanding this dynamic character of reality that defines context problems, starting points for instructional sequences will often link up with everyday-life experience of students. It is precisely the connection with velocity and distance that offers the students the means to reason and act in a meaningful manner from the start.

## NOTES

[1] Following Lehrer and Romberg (1996) we use the term inscriptions to refer to symbolizations that have physical form.

[2] Some mathematicians argue that Oresme's proof of Merton's rule is not allowed. First he should have defined velocity as a differential quotient and then deduced the distance traversed by graphical integration. The mathematical historian Dijksterhuis discusses this and defends Oresme by stating: It is a situation which occurs regularly in the history of mathematics: mathematical concepts are often – maybe even: usually – used intuitively for a long time before they can be described accurately, and fundamental theorems are understood intuitively before they are proven (Dijksterhuis, 1980).

[3] The sequence is called Sum & Difference, Distance & Speed (Som & Verschil, Afstand & Snelheid – Kindt, 1997). One of the chapters of this sequence is based on a text of Polya on the history of Galileo (Polya, 1963). Our description does not correspond with the existing unit in every detail, but anticipates certain revisions.

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KOENO GRAVEMEIJER and MICHIEL DOORMAN
*Freudenthal Institute,*
*Utrecht University,*
*The Netherlands*
*E-mail: koeno@fi.ruu.nl*