Supporting Rigorous Mathematics Teaching and Learning

The Instructional Tasks Matter: Analyzing the Demand of Instructional Tasks

PARTICIPANT HANDOUT

Tennessee Department of Education

High School Mathematics

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Rationale – Comparing Mathematical Tasks

Tasks form the basis for students' opportunities to learn what mathematics is and how one does it, yet not all tasks afford the same levels and opportunities for student thinking. [They] are central to students' learning, shaping not only their opportunity to learn but also their view of the subject matter.

Adding It Up, National Research Council, p. 335, 2001

By analyzing two tasks that are mathematically similar, teachers will begin to differentiate between tasks that require thinking and reasoning and those that require the application of previously learned rules and procedures.

Learning Goals and Activities

Participants will:

- compare mathematical tasks to determine the demand of the tasks; and
- identify the Common Core State Standards (CCSS) for Mathematical Content and the Common Core State Standards (CCSS) for Mathematical Practice addressed by each of the tasks.

Comparing the Cognitive Demand of Mathematical Tasks

What are the similarities and differences between the two tasks?

- Solve It Task
- Softball Task

Similarities	Differences

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Comparing Two Mathematical Tasks

How are the two tasks the same and how are they different?

The Solve It Task

Use the quadratic formula to solve the following equations.

1. $3x^2 + 5x – 2 = 0$
2. $2x^2 - 7x = 4$
3. $-x^2 + 9x + 1 = 0$
4. $x - 4x^2 = -9$

The Softball Task

Kara hits a softball straight up at a speed of 70 ft/s. Her bat contacts the ball at a height of 3 ft above the ground. The equation relating height in meters, y, and time in seconds, x, is

$$y = -16x^2 + 70x + 3$$

How long will it take until the ball hits the ground? Explain your reasoning in words.

Do the differences between the Solve It Task and the Softball Task impact students' opportunities to learn mathematical concepts and practices? Why or why not?

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The Mathematical Task Analysis Guide

Lower-Level Demands	Higher-Level Demands
Memorization Tasks • Involve either producing previously learned facts, rules, formulae, or definitions OR committing facts, rules, formulae, or definitions to memory. • Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. • Are not ambiguous – such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated. • Have no connection to the concepts or meaning that underlie the facts, rules, formulae, or definitions being learned or reproduced.	Procedures With Connections Tasks • Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • Suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning. • Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.
Procedures Without Connections Tasks • Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task. • Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • Have no connection to the concepts or meaning that underlie the procedure being used. • Are focused on producing correct answers rather than developing mathematical understanding. • Require no explanations, or explanations that focus solely on describing the procedure that was used.	Doing Mathematics Tasks • Requires complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example). • Requires students to explore and to understand the nature of mathematical concepts, processes, or relationships. • Demands self-monitoring or self-regulation of one's own cognitive processes. • Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • Requires students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • Requires considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

Stein, Smith, Henningsen, & Silver (2000). Implementing standards-based mathematics instruction: A casebook for professional development, p. 16. New York: Teachers College Press

The Common Core State Standards

Examine the CCSS

- for Mathematical Practice
- for Mathematical Content

- Which standards does each task address?

- What kind of engagement does each task require?

- Which Mathematical Practice Standards will students have opportunities to use?

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Reasoning with Equations and Inequalities — A-REI

Solve equations and inequalities in one variable

A-REI.B.4 Solve quadratic equations in one variable.

A-REI.B.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Represent and solve equations and inequalities graphically

A-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Creating Equations — A-CED

Create equations that describe numbers or relationships

A-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*

Interpreting Functions — F-IF

Interpret functions that arise in applications in terms of the context

F-IF.B.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*

F-IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

Analyze functions using different representations

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

F-IF.C.7a Graph linear and quadratic functions and show intercepts, maxima, and minima.

The Cognitive Demand of Tasks

- Working individually, use the TAG to determine if tasks A – P are high- or low-level tasks.
- Identify and record the characteristics on the TAG that best describe the cognitive demand of each task.
- Identify the CCSS for Mathematical Practice that the written task requires students to use.
- Share your categorization in pairs or trios. Be prepared to justify your conclusions using the TAG and the practices.

Task	Categorization	The CCSS for Content and for Mathematical Practice
A		
B		
C		
D		

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Task	Categorization	The CCSS for Content and for Mathematical Practice
E		
F		
G		
H		

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The Cognitive Demand of Tasks

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Task	Categorization	The CCSS for Content and for Mathematical Practice
I		
J		
K		
L		

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The Cognitive Demand of Tasks

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Task	Categorization	The CCSS for Content and for Mathematical Practice
M		
N		
O		
P		

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Quadratic Equations A – D

Task A	Task B
Use the quadratic formula to solve the equation. $16x^2 = 70x + 3$	Consider the general quadratic equation $ax^2 + bx + c = 0$ Assume that *a, b*, and *c* are integers. 1. Give some conditions of *a, b*, and *c* such that the equation has exactly two rational roots. 2. Find values for *a, b,* and *c* that produce exactly 2 rational roots.
Task C	**Task D**
What formula can be used to solve the equation $ax^2 + bx + c = 0$? State the name of the formula and write it symbolically.	Kara hits a softball straight up at a speed of 70 ft./s. Her bat contacts the ball at a height of 3 ft. above the ground. The equation relating height in meters, *y*, and time in seconds, *x*, is $y = -16x^2 + 70x + 3$ How long will it take until the ball hits the ground? Explain your reasoning in words.

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Systems of Equations E - H

TASK E

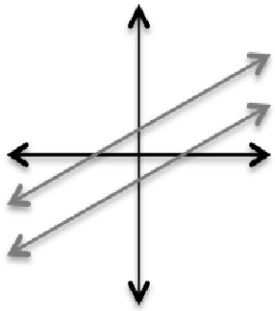
The table shows the price in dollars that long-distance companies A and B charge per minute that you are on the phone.

Time (in minutes)	0	30	60	90	120
Company A	5.00	6.20	7.40	8.60	9.80
Company B	2.00	5.00	8.00	11.00	14.00

Write the equations that will give the cost for any number of minutes on the phone for Company A and Company B. How much time time per month would you have to talk on the phone before subscribing to Company A would save you money?

TASK F

How many solutions does this system of equations have?



a) One
b) Zero
c) Infinite

TASK G

Long-distance Company A charges a base rate of $5 per month, plus 4 cents per minute that you are on the phone. Long-distance Company B charges a base rate of only $2 per month, but they charge 10 cents per minute used. How much time per month would you have to talk on the phone before subscribing to Company A would save you money?

Achieve, Inc., 2002

TASK H

What is the solution of the system?

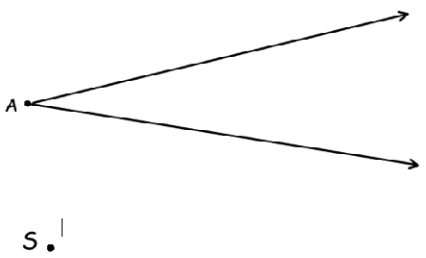
$$-4x - y = 2$$
$$7x + 2y = -5$$

a. (2, –6)
b. (–1, 6)
c. (1, –6)

Constructing Angles and Triangles I - L

Task I

Using your compass and straightedge, follow the instructions for Example 2 on page ## to construct $\angle S \cong \angle A$ below.

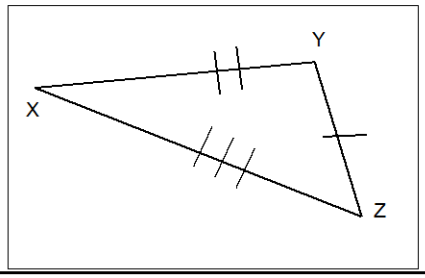


Which triangle congruence postulate can you use to explain why we can conclude that $\angle S \cong \angle A$?

Task J

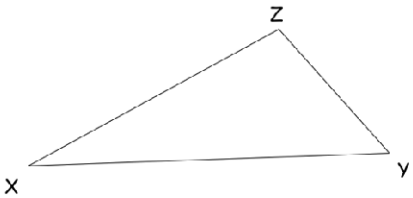
How can triangle $\triangle XYZ$ be classified? Circle all that apply.

a) Equilateral triangle
b) Scalene triangle
c) Obtuse triangle
d) Right triangle



Task K

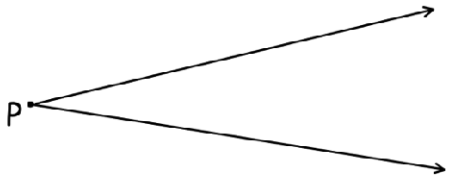
Using your compass and straightedge, work together with your teammates to figure out a way to construct $\triangle ABC$ so that $\triangle ABC \cong \triangle XYZ$. **Hint:** you can use any of the triangle congruence postulates as a basis for your construction.



Describe your process and explain how the triangle congruence postulates, and your construction, verify that the two triangles are congruent.

Task L

Using your compass and straightedge, follow the instructions for Example 2 on page ## to construct an angle congruent to angle P.



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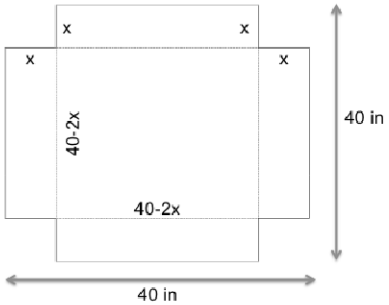
Polynomial Functions M - P

Task M	Task N
The Hold-It Container Company has a large supply of 40-inch squares of cardboard. They want to design an open-top rectangular box by removing squares of side length x from each corner of a 40-inch square piece of cardboard as shown below.  a) Express the volume of the box as a function of x. b) Graph the volume function and determine the x value which will produce the maximum volume.	The Hold-It Container Company has a large supply of 40-inch squares of cardboard. They want to design an open-top rectangular box by removing squares of side length x from each corner of a 40-inch square piece of cardboard. What is the maximum volume this box can have? Use equations, graphs, and/ or words to explain your answer.
Task O	**Task P**
Consider the following polynomial. $y = 4x^3 - 160x^2 + 1600x$ Write the degree of the polynomial. _____ Write the leading coefficient. _____ Write the number of terms. _____	Use your graphing calculator to graph the function $y = 4x^3 - 160x^2 + 1600x$ Determine the maximum and minimum value when $0 \le x \le 10$

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References

Smith, M. S., Stein, M. K., Arbaugh, F., Brown, C. A., & Mossgrove, J. (2004). Characterizing the cognitive demands of mathematical tasks: A task-sorting activity. In G. W. Bright and R. N. Rubenstein (Eds.), Professional Development Guidebook for perspectives on the teaching of mathematics: Companion to the sixty-sixth yearbook. Reston, VA: National Council of Teachers of Mathematics.

Smith, M. S. & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. Mathematics Teaching in the Middle School, 3 (5), 344-350.

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Comparing Two Mathematical Tasks

Compare the similarities and differences between the two tasks:

- The Solve It Task
- The Softball Task

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The Solve It Task

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Comparing Two Mathematical Tasks

- Do the differences between the Solve It Task and the Softball Task impact students' opportunities to learn mathematical concepts and practices?
- Why or why not?

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The Common Core State Standards

- Examine the CCSS
  - for Mathematical Practice
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- Which standards does each task address?
- What kind of engagement does each task require?
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The CCSS for Mathematical Content

CCSS Algebra Conceptual Category

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Solve equations and inequalities in one variable

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Common Core State Standards, 2010

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Analyze functions using different representations

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

F-IF.C.7a Graph linear and quadratic functions and show intercepts, maxima, and minima.

Common Core State Standards, 2010

Evidence of Standards for Mathematical Practice

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The Softball Task	The Solve It Task
Make sense of problems and persevere in solving them	
Students must find a way to solve the problem.	Students can recall from memory the quadratic formula. They do not have to "make sense" of the problem.
Reason abstractly and quantitatively	
Students must figure out how to use the equation and relate it to the variables in the context to answer the question about "how long will it take".	Students are not required to make a visual representation, or to make connections among the representations.
Construct viable arguments	
Students explain how they determined the amount of time.	No explanation is required.

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Linking to Research/Literature: The QUASAR Project

Not all tasks are created equal, and different tasks will provoke different levels and kinds of student thinking.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000).
Implementing standards-based mathematics instruction: A casebook for professional development
New York: Teachers College Press.

Linking to Research/Literature

There is no decision that teachers make that has a greater impact on students' opportunities to learn and on their perceptions about what mathematics is than the selection or creation of the tasks with which the teacher engages students in studying mathematics.

Lappan & Briars, 1995

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Characterizing Tasks in Terms of Cognitive Demands

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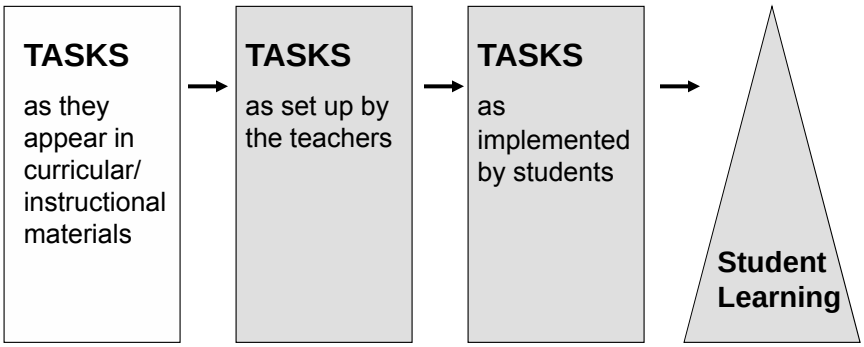
Linking to Research/Literature: The QUASAR Project

The Mathematical Tasks Framework



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Linking to Research/Literature: The QUASAR Project

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- Low-Level Tasks
  - The Solve It Task
- High-Level Tasks
  - The Softball Task

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Linking to Research/Literature: The QUASAR Project

- Low-Level Tasks
  - memorization
  - procedures without connections (e.g., Solve It Task)
- High-Level Tasks
  - procedures with connections
  - doing mathematics (e.g., the Softball Task)

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The Task Analysis Guide

Research has identified characteristics related to each of the categories on the Mathematical Task Analysis Guide (TAG).

Study the *Doing Mathematics* category.

How do the characteristics that we identified when discussing the Softball Task relate to those on the TAG?

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The Cognitive Demand of Tasks

- Working individually, use the TAG to determine if tasks A – P are high- or low-level tasks.
- Identify and record the characteristics on the TAG that best describe the cognitive demand of each task.
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Identifying High-level Tasks

Which of the four tasks are considered to have a high level of cognitive demand and why?

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Relating the Cognitive Demand of Tasks to the Mathematical Practices

What relationships do you notice between the cognitive demand of the written tasks and the CCSS for Mathematical Practice listed for each task?

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Linking to Research/Literature: The QUASAR Project

If we want students to develop the capacity to think, reason, and problem solve then we need to start with high-level, cognitively complex tasks.

Stein, M. K. & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. Educational Research and Evaluation, 2 (4), 50-80.

Linking to Research

Tasks are central to students' learning, shaping not only their opportunity to learn but also their view of the subject matter.

National Research Council (2001). J. Kilpatrick, J. Swafford, & B. Findell, (Eds.). Adding it up: Helping children learn mathematics, p. 335. Washington, DC: National Academy Press.

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